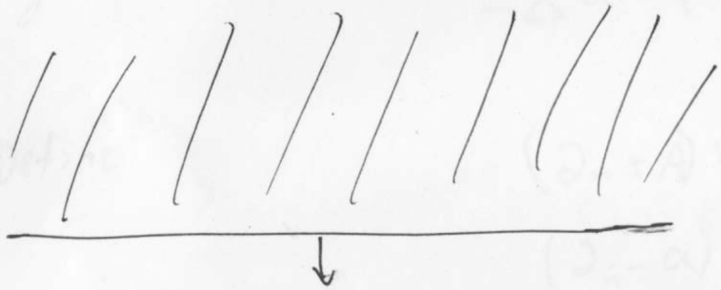
DN Operator

$$\begin{cases} -\Delta u = 0 \\ u|_{\partial\Omega} = f \in C^\infty ? \end{cases} \qquad T_{DN} f = \partial_n f$$

upper half plane

$$\begin{cases} u(x,0) = f(x), \ x \in \mathbb{R}^d \\ -\Delta u = 0 \end{cases}$$

Claim. $T_{DN} f = (-\Delta)^{1/2} f$.



- Define $Tf(x) = -u_y(x,0)$

  then $-u_y(x,y)$ solves

$$\begin{cases} v(x,0) = Tf \\ -\Delta v = 0 \end{cases}$$

- $T^2 f = T(Tf) = T(-u_y(x,0)) = u_{yy}(x,0) = -\Delta_x f$

- T is positive: By Green's formula:

$$\langle Tf, f\rangle = \int_{\mathbb{R}^d} [redacted] (Tf) f = \int_{\mathbb{R}^d} -u_y(x,0) f(x) = \int_{\mathbb{R}^{d+1}} |\nabla u|^2 + \cancel{u\Delta u}^{\,0}$$

≥ 0

Basically, true in general, symbol for $-\Delta$ is $|\xi|^2$.

On Riemanian manifold (M, g), have basically principal symbol of T_{DN} is (thesis by Jan Möllers)

$$a(x,\xi) = \sqrt{\sum_{j,k=1}^{n-1} g^{jk}(x)\xi^j \xi^k}.$$

$$\left(T_a f(x) = \int a(x,\xi)\hat{f}(\xi) e^{ix\xi}\, d\xi \right)$$

This can be seen by "factoring" the Laplacian. (or its symbol).

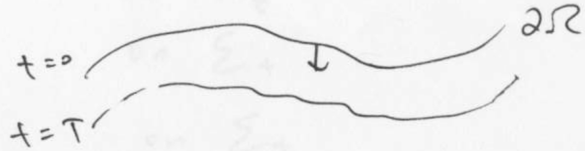
Say can factor

$$-\Delta u = (\partial_n - \underbrace{a}_{\text{tangential}})(\partial_n + \underbrace{A}_{\text{tangential}})u + \underbrace{R}_{\text{smoothing}} u .$$

Define

$$(\partial_n + A)u = v$$

$$(\partial_n - a)v = -Ru \in C^\infty$$



Elliptic regularity $\Rightarrow u \in C^\infty$ in Ω°.

$$v(T) = (\partial_n + A)u(T) \in C^\infty$$

$$\frac{dv}{dt} - av \in C^\infty$$

Solve backwards, reg. for heat eqn

$$\Rightarrow v(0) \in C^\infty$$

Finally,

$$\partial_n f = \frac{du}{dt}(0)$$

$$= -Au(0) - v(0)$$

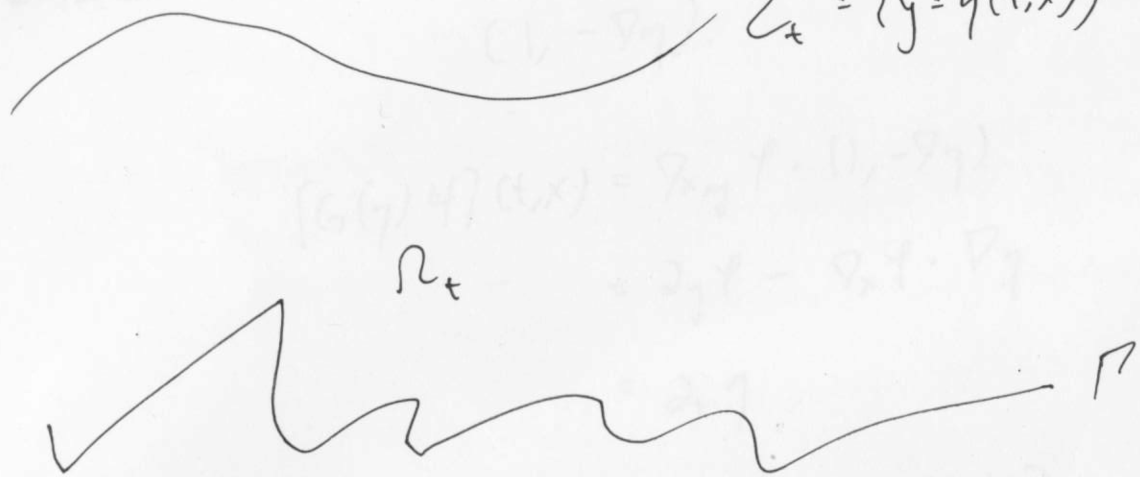
$$= -A(0)u - v(0)$$

DN for Water Wave Equation

(On WWE with Surface Tension by Alazard, Burq, Zuily)

Setup $\{(t,x,y) \in [0,T) \times \mathbb{R}^d \times \mathbb{R} : (x,y) \in \Omega_t\}$.

$\Sigma_t = \{y = \eta(t,x)\}$



$$\begin{cases} \Delta_x \varphi + \partial_y^2 \varphi = 0 & \text{in } \Omega_t \\ \partial_t \eta = \partial_t \varphi - \nabla \eta \cdot \nabla \varphi & \text{on } \Sigma_t \\ \partial_t \varphi = -g\eta + \kappa H(\eta) - \frac{1}{2}|\nabla \varphi|^2 - \frac{1}{2}|\partial_t \varphi|^2 & \text{on } \Sigma_t \\ \partial_n \varphi = 0 & \text{on } \Gamma \end{cases}$$

Defn. Let $\psi(t,x) = \varphi(t,x,\eta(t,x))$

DN-operator: $G(\eta)\psi = \sqrt{1 + |\nabla \eta|^2} \cdot \partial_n \varphi \big|_{y=\eta(t,x)}$

Note that
$$\Sigma_t = \{(x,y) : y - \eta(x) = 0\}.$$

So outward normal is
$$(1, -\nabla\eta).$$

Hence
$$\begin{aligned}[G(\eta)\psi](t,x) &= \nabla_{x,y}\varphi \cdot (1, -\nabla\eta) \\ &= \partial_y \varphi - \nabla_x\varphi \cdot \nabla\eta \\ &= \partial_t \eta .\end{aligned}$$

Also,
$$\left.\begin{aligned}\partial_y\varphi &= [G(\eta)\psi] + \nabla\varphi\cdot\nabla\eta \\ \nabla_x\varphi &= \nabla\psi - (\partial_y\varphi)\nabla\eta\end{aligned}\right\} \Rightarrow \partial_y\varphi = \underbrace{\frac{[G(\eta)\psi] + \nabla\eta\cdot\nabla\psi}{1+|\nabla\eta|^2}}_{:= B}$$

Thus we have
$$\begin{aligned}\partial_y\varphi &:= B \\ \nabla_x\varphi &:= \nabla\psi - B\nabla\eta := V\end{aligned}$$

<u>Rmk</u>. Later the unknown $\psi - T_B \eta$ will turn out to be important.

Further, we see now that (η,ψ) solve the Cauchy problem

$$\begin{cases} \eta(0,x)=\eta_0, \quad \psi(0,x)=\psi_0 \\ \partial_t \eta - [G(\eta)\psi] = 0 \\ \partial_t \psi + g\eta - H(\eta) + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\dfrac{G(\eta)\psi + \nabla\eta\cdot\nabla\psi}{1+|\nabla\eta|^2} = 0. \end{cases}$$

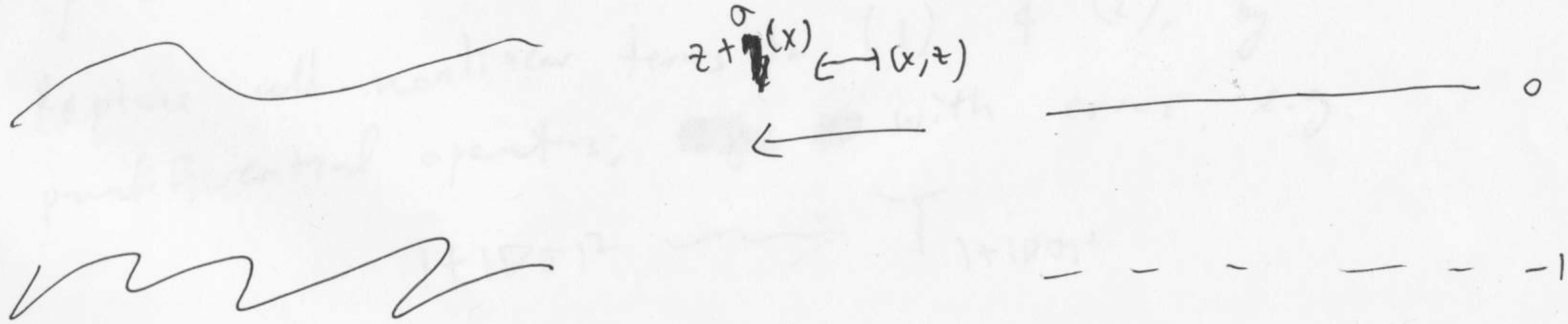
Idea: Paralinearize $\rightsquigarrow$ reduce to simple hyperbolic form.
(paradifferential symmetrizer).

- Existence, uniqueness, regularity $\quad d\geq 1,\ s>2+d/2.$ $(\eta_0,\psi_0)\in H^{s+1/2}\times H^s \Rightarrow (\eta,\psi)\in H^{s+1/2}\times H^s$
- Smoothing effect.

Full paralinearization: (will follow "Paral. of DN operator, and regularity of 3D WW," by Alazard, Métivier)

$$G(\sigma)\psi = Op(\lambda_\sigma)\psi + \underbrace{B(\sigma,\psi)}_{\text{explicit}} + \underbrace{R(\sigma,\psi)}_{\partial x \text{ more regular than } \sigma,\psi}.$$

Change of variables.



Define

$$v(x,z) = \varphi(x, z+\sigma(x))$$

$$v|_{z=0} = \varphi|_{y=\sigma(x)} = \psi .$$

and

$$\boxed{(1+|\nabla\sigma|^2)\,\partial_z^2 v + \Delta v - 2\nabla\sigma\cdot\nabla\partial_z v - \partial_z v\,\Delta\sigma = 0} \qquad (1)$$

Since

$$\partial_z v = \partial_y \varphi$$

$$\nabla_x v = \nabla_x \varphi + (\partial_y\varphi)\nabla\sigma$$

$$G(\sigma)\psi = \partial_y\varphi - \nabla_x\varphi\cdot\nabla\sigma$$

$$= \partial_z v - \left(\nabla_x v - (\partial_z v)\nabla\sigma\right)\cdot\nabla\sigma$$

$$\boxed{G(\sigma)\psi = (1+|\nabla\sigma|^2)\,\partial_z v - \nabla\sigma\cdot\nabla v\,\big|_{z=0}} \qquad (2)$$

Idea for paralinearization:

Replace all nonlinear terms in (1) & (2), by paradifferential operators, [redacted] with error, e.g.

$$1+|\nabla\sigma|^2 \rightsquigarrow T_{1+|\nabla\sigma|^2}$$