Paradiff. Operators (Métivier)

Want to consider symbols with [redacted] $a(x,\xi)$ with limited spatial smoothness.

$$\Gamma^m_\rho = \{ a(x,\xi) : \text{locally bdd}, \ C^\infty \text{ wrt } \xi \text{ for } \xi \neq 0.$$

$$\forall \alpha : \quad x \longmapsto \partial^\alpha_\xi a(x,\xi) \text{ in } C^\rho \quad (\text{or } W^{\rho,\infty}).$$

and $\quad \forall |\xi| \geq \frac{1}{2},$

$$\| \partial^\alpha_\xi a(x,\xi) \|_{C^\rho} \lesssim C(1+|\xi|)^{\underbrace{m}_{\text{order of operator}}-|\alpha|}$$

Will smooth out $a(x,\xi)$, i.e. consider spectral cutoff fcn $\chi \in C^\infty$.

[redacted] $\exists\, 0 < \varepsilon_1 < \varepsilon_2$

$$\chi(\theta,\xi) = 1 \qquad |\theta| \leq \varepsilon_1 |\xi|$$

$$\chi(\theta,\xi) = 0 \qquad |\theta| \geq \varepsilon_2 |\xi|$$

[redacted] So consider

$$a(\cdot,\xi) \longmapsto \check{\chi}(\cdot,\xi) * a(\cdot,\xi)$$

i.e.

$$\hat{a}(\eta,\xi) \longmapsto \chi(\eta,\xi)\, \hat{a}(\eta,\xi).$$

with associated operator

$$\widehat{T_a u}(\eta) = \frac{1}{(\sqrt{2\pi})^d} \int \chi(\eta-\xi,\xi)\, \hat{a}(\eta-\xi,\xi)\, \hat{u}(\xi)\, d\xi$$

Further, need to consider $a(x,\xi)$ not smooth at $\xi = 0$. (e.g. DN operator)

so introduce $\psi \in C^\infty$, s.t.

$$\psi(\xi) = 0 \qquad |\xi| \leq 1$$

$$\psi(\xi) = 1 \qquad |\xi| \geq 2.$$

and

$$\widehat{T_a u}(\eta) = \frac{1}{(\sqrt{2\pi})^d} \int \chi(\eta-\xi,\xi)\, \underbrace{\tilde{a}(\eta-\xi,\xi)}_{\hat{a}(\eta-\xi,\xi)} \psi(\xi)\, \hat{u}(\xi)\, d\xi$$

Defn. $\operatorname{ord}(T) \leq m$ if $T : H^{s+m} \to H^s$ bdd.

Thm. $a \in \Gamma_0^m \Rightarrow T_a$ order $\leq m$.

Composition. $a \in \Gamma_\rho^m$, $b \in \Gamma_\rho^{m'}$, then

$$T_a T_b - T_{a\#b} \quad \text{order } m+m'-\rho$$

where

$$a\#b = \sum_{|\alpha|<\rho} \underbrace{\frac{1}{i^\alpha \alpha!} \partial_\xi^\alpha a(x,\xi)\, \partial_x^\alpha b(x,\xi)}_{\substack{\text{Taylor expansion,} \\ \text{and } D_x \leadsto i\xi_x}} \in \sum_{j<\rho} \Gamma_{\rho-j}^{m+m'-j}$$

Note the leading term is just ab.

Special Cases: Fourier multiplier ($a(\xi)$ indep. of x)

Paraproduct ($a(x)$ indep. of ξ).

For our purposes, we use paraproducts. e.g., $\sigma(x) \rightsquigarrow T_\sigma$.

One easy lemma:

<u>Lemma</u>. $a, b \in H^\alpha$, $\alpha > d/2$, $\alpha - d/2 \notin \mathbb{N}$. Then

$$T_a T_b - T_{ab} \quad \text{order} \leq -\lfloor \alpha - \tfrac{d}{2} \rfloor$$

<u>Pf</u>. By Sobolev embedding, a, b bounded, $a, b \in C^{\lfloor \alpha - \frac{d}{2} \rfloor}$.

So

$$a, b \in \Gamma^0_{\lfloor \alpha - \frac{d}{2} \rfloor}$$

By composition lemma,

$$T_a T_b - T_{a\#b} = T_a T_b - T_{a\#b}$$

$$\text{order} \quad 0 + 0 - \lfloor \alpha - \tfrac{d}{2} \rfloor$$

$\square$

More involved; have paralinearization lemma:

<u>Lemma</u>. • $a \in H^\alpha$, $\alpha > d/2$, $F \in C^\infty$.

$$F(a) - T_{F'(a)} \in H^{2\alpha - d/2}$$

• $a \in H^\alpha$, $b \in H^\beta$, $\alpha + \beta > 0$,

$$\boxed{ab - T_a b - T_b a \in H^{\alpha + \beta - d/2}}$$

Lemma. $\alpha \in \mathbb{R}$, $\beta < d/2$,

$$a \in H^\alpha, \quad b \in H^\beta \implies T_b a \in H^{\alpha+\beta-d/2}$$

We also need:

Lemma. $a \in H^\alpha$, $b \in H^\beta \implies ab \in H^{[(\alpha\wedge\beta)\wedge(\alpha+\beta-d/2) \geq \gamma]}$

$ab \in H^\gamma$ for $\gamma \leq (\alpha\wedge\beta)\wedge(\alpha+\beta-d/2)$

and the following result on regularity for heat eqn:

Lemma. $\varepsilon \in [0,1)$, $a \in \Gamma^1_{1+\varepsilon}$, $b \in \Gamma^0_0$. $\exists c > 0$ s.t.

$$\forall (x,\xi), \quad \operatorname{Re} a(x,\xi) \geq c|\xi|.$$

Then if $u \in C^1([-1,0]; H^{-\infty})$ solves

$$\partial_t u + T_a u = T_b u + f$$

for $f \in C^0([-1,0]; H^s)$, then

$$u(0) \in H^{s+\varepsilon}$$

maybe mention later