I. Introduction and Existence of Phase Transition

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Setting. Let $0 \leq p \leq 1$. For now on infinite square lattice $\mathbb{Z}^d$, $\omega \subset \mathbb{E}$ (set of edges)

- Probability of each *configuration* ω is

$$\mathbb{P}_p(\omega) = p^{|\omega|} \cdot (1-p)^{|\omega^c|} \quad (|\omega| = \text{cardinality of } \omega)$$

- Alternatively, each bond is *independently open* with probability p and $\mathbb{P}_p$ is product measure on the σ–algebra generated by *cylinder events*, i.e., events concerning a *finite* number of edges.
- We will also think of $\omega \subset \mathbb{E}$ as being colored *blue*.
- Later will consider other lattices and will focus on $d = 2$.

Definition. Let $x \in \mathbb{Z}^d$ be a site, then

$$\mathcal{C}_\omega(x) = \{y \in \mathbb{Z}^d : \{x \leftrightsquigarrow y\} \ \text{ occurs}\}$$

is the *open* cluster containing x. It is a random set.

[picture of a cluster]

Central Question. When is $|\mathcal{C}(0)| = \infty$ with *positive* probability?

Definition. Let

$$p_c = \inf\{p : \theta(p) > 0\},$$

where

$$\theta(p) := \mathbb{P}_p(|\mathcal{C}(0)| = \infty).$$

- Is $p_c < 1$? (clear that $p_c = 1$ for $d = 1$)
- Is $p_c > 0$? (think long–range interactions)

More Questions.

- Value of p_c?
- Behavior near p_c?

$$\theta(p) \sim (p - p_c)^\beta$$

(*power law* behavior... β is *critical exponent*...)

- Relevance of choice of dimension and lattice (universality)?
- Large scale properties under introduction of mild dependence (correlations)?

Existence of Phase Transition. We now establish that p_c is non–trivial for $d \geq 2$, i.e., non–trivial phase transition. The model is

subcritical, *critical*, *supercritical* when $p < p_c$, $p = p_c$, $p > p_c$, respectively.

Theorem. *If $d \geq 2$, then $0 < p_c(\mathbb{Z}^d) < 1$.*

Proof. Will show

$$\frac{1}{\mu_d} \leq p_c(\mathbb{Z}^d) \leq 1 - \frac{1}{\mu(2)}, \quad \text{for } d \geq 2,$$

where μ_d is the *connective constant* for $\mathbb{Z}^d$ (for SAW). Sufficient since $\mathbb{Z}^d \hookrightarrow \mathbb{Z}^{d+1}$, so

$$p_c(\mathbb{Z}^d) \leq p_c(\mathbb{Z}^{d+1}).$$

(Aside: Let $C_d(n)$ be the number of SAW starting at 0 in $\mathbb{Z}^d$ of length n. Then

$$\mu_d = \lim_{n\to\infty} (C_d(n))^{1/n}.$$

- Since a SAW of length n is a concatenation of a walk of length k and length $n-k$,

$$C_d(n) \leq C_d(k) \cdot C_d(n-k).$$

Therefore,

$$\log C_d(n) \leq \log C_d(k) + \log C_d(n-k),$$

so the sequence $\{a_n\} := \{\log C_d(n)\}$ is *subadditive* and so

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}.$$

Indeed, it is sufficient to see that $\limsup_{n\to\infty}(a_n/n) \leq \inf_{n\geq 1}(a_n/n)$: Write $n = km+\ell$, then

$$a_n \leq k a_m + a_\ell,$$

so that

$$\frac{a_n}{n} \leq \frac{km}{km+\ell} \cdot \frac{a_m}{m} + \frac{a_\ell}{n}.$$

This gives existence of $\lim_{n\to\infty} \frac{\log C_d(n)}{n}$, so

$$\mu_d = \lim_{n\to\infty} \exp\left(\frac{\log(C_d(n))}{n}\right).$$

- Note also that

$$C_d(n) \leq 2^d(2^d-1)^{n-1},$$

so that

$$(C_d(n))^{1/n} \leq (2^d - 1) \quad \Longrightarrow \quad \frac{1}{\mu_d} > 0.$$

~~~~~~~)

Claim. $p_c(\mathbb{Z}^d) \geq \frac{1}{\mu_d}$.

Let $N(n)$ denote the number of *open* paths of length n starting at the origin, then

$$\begin{aligned}
\theta(p) &\leq \mathbb{P}_p(N(n) \geq 1) \\
&\leq \mathbb{E}_p(N(n)) \quad (= \sum_{k=1}^{\infty} k \cdot \mathbb{P}_p(N(n)=k)) \\
&\leq C_d(n) \cdot p^n \\
&= \left((C_d(n))^{1/n} \cdot p\right)^n
\end{aligned}$$
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for all n, so taking $n \to \infty$ we see that $\theta(p) = 0$ unless

$$(C_d(n))^{1/n} \cdot p \geq 1 \quad \Longrightarrow \quad p \geq \frac{1}{\inf_{n\geq 1}(C_d(n))^{1/n}} = \frac{1}{\mu_d}.$$

Claim. $p_c(\mathbb{Z}^d) \leq 1 - \frac{1}{\mu(2)}$ for $d \geq 2$.

Here we use a *Peierls argument*, based on *2D duality*: Each *finite* cluster is surrounded by a *circuit* of the *dual lattice*

$$\mathbb{Z}_2^* = \mathbb{Z}_2 + (1/2, 1/2).$$

By duality we mean that given a configuration of open edges ω on the *direct lattice* $\mathbb{Z}^2$, the dual configuration ω^* is given by

$$\omega^* = \{\langle x^*, y^*\rangle : \langle x, y\rangle \notin \omega.\}$$

Here $\langle x, y\rangle$ denotes a bond with endpoints x, y and $\langle x^*, y^*\rangle \in \mathbb{Z}_2^*$ denotes the corresponding dual bond, i.e., the bond which *crosses* $\langle x, y\rangle$. We sometimes think of ω^* as being *yellow*.

[picture of lattice and its dual with dual circuit surrounding direct finite clusters]

Let $M(n)$ be the number of dual circuits of length n. First note that

$$M(n) \leq n \cdot C_2(n-1).$$

Indeed, each such circuit must pass through some vertex

$$(k + 1/2, 1/2) \qquad \text{for} \qquad 0 \leq k < n$$

and deleting one of the edges containing this vertex what remains of the circuit is a SAW starting at this vertex of length $n-1$. We can estimate as before that

$$\begin{aligned} 1-\theta(p) &= \mathbb{P}_p(|C(0)|<\infty) \\ &= \sum_n \mathbb{P}_p(M(n)\geq 1) \\ &\leq \sum_n \mathbb{E}_p(M(n)) \\ &\leq \sum_n nC_2(n-1)\cdot(1-p)^n \quad (=\sum_n n\cdot(C_2(n-1)^{1/n}\cdot(1-p))^n), \end{aligned}$$

so the last sum certainly converges if $1-p<\frac{1}{\mu_2}$.

Now we perform a refinement which will yield the conclusion. Let B_m denote a box of side–length m and

$$\begin{aligned} F_m &= \{\omega : \omega^* \text{ contains a circuit surrounding } B_m\} \\ G_m &= \{\omega : \text{all bonds of } B_m \text{ are occupied}\}. \end{aligned}$$

Note that

- $$F_m^c = \{\omega : \exists \text{ a vertex on } \partial B_m \text{ which is connected to } \infty\}$$

  so

  $$F_m^c \cap G_m \quad\subseteq\quad \{|C(0)|=\infty\}.$$

- F_m and G_m are *independent* since they concern the state of *disjoint* edges.

[picture of F_m^c]

We may now estimate $\mathbb{P}_p(F_m)$ as before, except now the sum starts at $n=4m$, i.e., the estimate is the *tail* of the series:

$$\begin{aligned}\mathbb{P}_p(F_m) &= \sum_{n=4m}^{\infty} \mathbb{P}_p(M(n) \geq 1) \\ &\leq \sum_{n=4m}^{\infty} nC_2(n-1)\cdot(1-p)^n.\end{aligned}$$

Therefore, given any $0 < \lambda < 1$, if $1 - p < \frac{1}{\mu_2}$ then we may choose m sufficiently large so

$$\mathbb{P}_p(F_m) < \lambda.$$

With such a choice of m we can now conclude:

$$\begin{aligned}\theta(p) &\geq \mathbb{P}_p(F_m^c \cap G_m) \\ &= \mathbb{P}_p(F_m^c)\cdot\mathbb{P}_p(G_m) \\ &\geq (1-\lambda)\cdot\mathbb{P}_p(G_m) \quad (= (1-\lambda)\cdot p^{2m(m+1)}) \\ &> 0.\end{aligned}$$

□

Definition. Let $\psi(p)$ denote the probability that there exists an infinite open cluster.

Since *i)* the model is translation invariant and *ii)* $\psi(p)$ is the probability of a *tail event* (it is the probability of an event which cannot be determined by any *finite* collection of edges) we have (by the *zero–one law*) that:

Corollary. The probability $\psi(p)$ satisfies

$$\psi(p) = \begin{cases} 0 & \text{if} \quad \theta(p) = 0 \\ 1 & \text{if} \quad \theta(p) > 0. \end{cases}$$

References.

1. *Percolation and disordered systems* by Geoffrey Grimmett. Lecture Notes in Mathematics (1997). Volume 1665/1997, 153-300.

2. *Percolation* by Geoffrey Grimmett. Springer–Verlag, Berlin Heidelberg (1999).