III. Rescaling and Scale Invariance

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We now restrict attention to $d = 2$ and collect here some scale invariant estimates.

RSW Estimates. The starting point is the so–called Russo–Seymour–Welsh estimate, which expresses the crossing probability of a (longer) rectangle in terms of that of a square. We start with the simplest version whose proof is due to Smirnov:

Lemma (RSW: Hexagonal tiling at criticality)**.** *Consider hexagonal tiling where hexagons are blue or yellow with probability* $\frac{1}{2}$*. If* $C(a,b) := \mathbb{P}(\mathcal{C}(a,b))$ *denotes the crossing probability of an* $a \times b$ *rectangle, then*

$$C(2a, b) \geq \frac{1}{4} \cdot C(a,b)^2.$$

Proof. Let $R(a,b)$ denote an $a \times b$ rectangle with bottom left corner equal to $(0,0)$. The starting point is to note that given any left right crossing γ of $R(a,b)$, and γ' being its (geometric) reflection across $y = b$, the domain below $\gamma \cup \gamma'$ intersected $R(a,b)$ forms a new domain $\mathcal{R}$ in which we can ask new crossing probability questions: we color $\gamma \cup [(a,0),(2a,0)] \cup [(2a,0),(2a,b)]$ blue and $\gamma' \cup [(0,a),(0,0)] \cup [(0,0),(a,0)]$ yellow, and consider the event $\mathcal{G}_b(\gamma)$ of a *blue* crossing from γ to J and the event $\mathcal{G}_y(\gamma)$ of a *yellow* crossing from γ' to J'

[picture of domain in question, colored appropriately...]

The key observation is that the events $\mathcal{G}_b$ and $\mathcal{G}_y$ are *dual* and *exhaustive*: there is a blue crossing if and only if there is no yellow crossing. Thus

$$\mathbb{P}(\mathcal{G}_b(\gamma)) + \mathbb{P}(\mathcal{G}_y(\gamma)) = 1.$$

Now by total symmetry of the regions in question *and* blue–yellow symmetry we conclude

$$\mathbb{P}(\mathcal{G}_b(\gamma)) = \mathbb{P}(\mathcal{G}_y(\gamma)) = \mathbb{P}(\mathcal{G}'_b(\gamma')) = 1/2.$$

(Here $'$ denotes the fact that we are envisioning the event taking place on the *right* rectangle.)

It is also clear that if both $\mathcal{G}_b(\gamma) \cap \{\gamma \text{ blue}\}$ and $\mathcal{G}_b(\gamma') \cap \{\gamma' \text{ blue}\}$ happen, then we must have a crossing of $R(2a, b)$:

$$\bigcup_{\gamma:[(0,0),(0,b)]\rightsquigarrow[(a,0),(a,b)];\quad \gamma':[(a,0),(a,b)]\rightsquigarrow[(2a,0),(2a,b)]} [\mathcal{G}_b(\gamma)\cap\{\gamma \text{ blue}\}]\cap[\mathcal{G}_b(\gamma')\cap\{\gamma' \text{ blue}\}] \subseteq \mathcal{C}(2a, b).$$

(Note that here γ' is not necessarily the reflection of γ.)

[picture of $\mathcal{G}_b(\gamma), \mathcal{G}_b(\gamma')$ forming crossing of $R(2a, b)$, with γ' *not* reflection of γ...]

It therefore remains to sum up over γ. Here we require the notion of *highest blue crossing:* given a fixed configuration ω, the left right crossings of any rectangle can be *partially ordered* according to the topological region above it and given any two crossings γ, γ' which are not comparable the boundary of the region above *both* of them defines a new crossing higher than each, which ensures the existence of a *maximum* element.

[picture illustrating lowest crossing and $\gamma \wedge \gamma'$...]

Two things are of interest about highest crossings:

- Let Π be the *random variable* denoting the highest crossing, then the events $\{\Pi = \gamma\}$ over all topological curves joining the left side to the right side of $R(a,b)$ *disjointly partitions* $\mathcal{C}(a,b)$:
$$\mathcal{C}(a,b) = \sum_{\gamma:[(0,0),(0,b)]\rightsquigarrow[(a,0),(a,b)]} \mathbb{P}(\Pi = \gamma).$$
- The event $\{\Pi = \gamma\}$ is *probabilistically independent* of the region *below* γ in $R(a,b)$. (The state of γ being the highest crossing is not disturbed by changing the state of any site above γ.)

From the second item we conclude that $\mathcal{G}_b(\gamma)$ is independent of the event $\{\Pi = \gamma\}$ *unless* γ hits the x–axis, i which case the *domain* of relevant crossing will shrink: it will be determined by the "last time" γ hits the x–axis and in case γ hits the *point* a we have $\mathbb{P}(\mathcal{G}_b(\gamma)) \equiv 1$. In any case it is always true that $\mathbb{P}(\mathcal{G}_b(\gamma) \mid \Pi = \gamma) \geq \frac{1}{2}$.

[picture of γ hitting x–axis forming non–trivial "small" domain; picture of γ hitting x–axis at point a, together with γ' forming a crossing of $R(2a,b)$...]

Summing up, we get that

$$\begin{aligned}\mathbb{P}(\mathcal{G}_b) &:= \mathbb{P}\left(\bigcup_{\gamma:[(0,0),(0,b)]\rightsquigarrow[(a,0),(a,b)]} \mathcal{G}_b(\gamma)\cap\{\gamma \text{ blue}\}\right)\\ &= \sum_\gamma \mathbb{P}(\mathcal{G}_b(\gamma)\cap\{\Pi=\gamma\})\\ &= \sum_\gamma \frac{1}{2}\cdot\mathbb{P}(\Pi=\gamma)\\ &= \frac{1}{2}\cdot C(a,b).\end{aligned}$$

Finally, by the FKG inequality,

$$C(2a,b) \geq \mathbb{P}(\mathcal{G}_b\cap\mathcal{G}_b') \geq \mathbb{P}(\mathcal{G}_b)^2 \geq \frac{1}{4}\cdot C(a,b).$$

□

The original argument due to Russo (1981) yields a bound for a crossing of $R(\frac{3}{2}\cdot a, a)$ in terms of crossing of the square $R(a,a)$ for *any* value of p; also the proof does *not* use *duality*. We will sketch this argument below. First a simple consequence of the FKG inequality:

Proposition ("Square root trick")**.** Let $A_1,\ldots,A_m$ be *increasing* events with *equal* probability. Then for $1\leq k\leq m$,

$$\mathbb{P}(A_k) \geq 1-\left\{1-\mathbb{P}\left(\cup_{i=1}^m A_i\right)\right\}^{1/m},$$

so in particular, with $m=2$, we have

$$\mathbb{P}(A_k)\geq 1-\sqrt{1-\mathbb{P}(A_1\cup A_2)},\quad k=1,2.$$

Proof. We note the set theoretic identity that

$$\left(\cup_{i=1}^m A_i\right)^c = \cap_{i=1}^m A_i^c,$$

with the A_i^c all being *decreasing* events so that the FKG inequality holds and we have:

$$\begin{aligned} 1-\mathbb{P}(\cup_{i=1}^m A_i) &= \mathbb{P}((\cup_{i=1}^m A_i)^c) \\ &= \mathbb{P}(\cap_{i=1}^m A_i^c) \\ &\geq \prod_{i=1}^m \mathbb{P}(A_i^c) \\ &= (1-\mathbb{P}(A_i))^m. \end{aligned}$$

□

Lemma (RSW)**.** *Set* $\tau = C(a,a)$*. Then*

$$C\left(\frac{3}{2}\cdot a, a\right) \geq (1-\sqrt{1-\tau})^3.$$

Proof. (Sketch). Consider the box $R(a,a)$ centered at the origin and $R(a,a)'$ which is $R(a,a)$ shifted *horizontally* by $\frac{1}{2}\cdot a$ to the *right*. It is clear that a crossing of $R(\frac{3}{2}\cdot a, a)$ would be accomplished if there are *horizontal* crossings of both boxes together with a *vertical* crossing of the $R(a,a)'$ to *join them*:

[picture of RSW argument: three crossings forming a crossing of larger rectangle with labels, π, α, β...]

Some thought reveals that some care is required to specify the beginning and ending points of these crossings; the precise prescription is as follows:

- $\Pi^{\pm}$: $\exists\pi$ an open *left right* crossing of $R(a,a)$ whose *last* intersection (oriented from left to right) with the *vertical midline* of $R(a,a)$ is *above* or *below* the x–axis;
- $A_\pi^{\pm}$: $\exists\alpha$ an open *path* from *top* of $R(a,a)'$ to π_r', π_r, respectively, where $\pi_r \subset \pi$ is the

last portion of π: from its last intersection with the vertical midline of $R(a,a)$ to the *right boundary* of $R(a,a)$ and π_r' is its reflection across the right boundary of $R(a,a)$;

- $B^{\pm}$: $\exists\beta$ an open *left right* crossing of $R(a,a)'$ starting on the top or bottom half of $R(a,a)'$;

we have then that

$$B^+ \cap \bigcup_{\pi}(A_\pi^- \cap \Pi^-) =: B^+ \cap G^- \subseteq \mathcal{C}\left(\frac{3}{2}\cdot a, a\right),$$

so that G^- is the event of a left right crossing of $B(a,a)$ together with a crossing connecting it to the top of $B(a,a)'$: an *increasing* event.

By symmetry of the regions in questions, it is clear that

$$\mathbb{P}(A_\pi^+) = \mathbb{P}(A_\pi^-), \quad \mathbb{P}(B^+) = \mathbb{P}(B^-),$$

so we have by the square root trick that

$$\mathbb{P}(B^+) \geq 1 - \sqrt{1-(B^+ \cup B^-)} = 1 - \sqrt{1-\tau},$$

so that by the FKG inequality

$$C\left(\frac{3}{2}\cdot a, a\right) \geq \mathbb{P}(B^+ \cap G^-) \geq (1-\sqrt{1-\tau})\cdot \mathbb{P}(G^-).$$

It remains to estimate $\mathbb{P}(G)$. Here as in the previous argument we will replace the event Π^- by L_Π^- which adds the additional requirement that π is the *lowest* left right crossing of $R(a,a)$, so that

$$G \supseteq \bigcup_{\pi} A_\pi^- \cap L_\pi^-.$$

We note again that supposing π only intersects the *vertical line of symmetry* once, A_π^- takes place above the region bounded by π and is therefore independent of L_π^- so that

$$\mathbb{P}_p(A_\pi^- \mid L_\pi^-) = \mathbb{P}_p(A_\pi^-).$$

In case multiple intersections occur, we observe that "*small regions*" would be formed in $B(a,a)'$ by π and since A_π^- is *independent* of the state of these regions, we may as well replace the conditioning (*without* changing the *conditional* probability) by the condition that the *entirety* of such regions (and their boundary, which are portions of π) are blue, and so by the FKG inequality, we would obtain

$$\mathbb{P}_p(A_\pi^- \mid L_\pi) = \mathbb{P}_p(A_\pi^- \mid \{\text{small regions blue}\}) \geq \mathbb{P}(A_\pi^-).$$

(The probability is in fact *higher* here since the crossing α has more possible landing points due to the "protrusions" caused by the multiple intersections.)

[picture of multiple intersection with vertical axis, with region "below" π shaded...]

Now it is clear that $\mathbb{P}(A_\pi^-) \leq \tau$ since the relevant *event* is contained in the event of a *top bottom* crossing of $R(a,a)'$. Therefore, summing up and using the square root trick to estimate $\mathbb{P}(A_\pi^-)$, we obtain

$$\mathbb{P}(G) \geq (1 - \sqrt{1-\tau}) \cdot \mathbb{P}(\Pi^-).$$

Applying the square root trick to Π^- yields the final factor of $1 - \sqrt{1-\tau}$. □

First Consequences.

We collect here some corollaries of the RSW estimate via iteration. What turns out to be useful (in addition to crossings of rectangles) are crossings in *annuli*:

Definition. Let $0 < a < b \in \mathbb{N}^+$ and denote by $A(a,b)$ the corresponding *annulus*:

$$A(a,b) = R(b,b) \setminus R(a,a).$$

By abuse of notation this may also denote the corresponding event of a *blue ring* inside the annulus or its *probability*.

Corollary. For all $p \in [0,1]$,

- $C_p(2a,a) \geq C_p(\frac{3}{2}a,a)^2 \cdot C_p(a,a)$;
- $C_p(3a,a) \geq C_p(2a,a)^2 \cdot C_p(a,a)$;
- $A_p(a,3a) \geq C_p(3a,a)^4$

Proof. These statements are proved by *pasting together* simpler events to form the event in question and estimating the probability by the FKG inequality. We have:

- A crossing of $R(2a,a)$ can be achieved as:

  [picture of achieving this event... left right crossings in $R(2a,a)$ and $R(2a,a)'$ which is its shift by a to the right... top bottom crossing in middle square $R(a,a)'$ to "stitch" the two together...]

- By increasing the scale this can be done exactly as the previous item.
- A ring in the relevant annulus can be guaranteed by the crossing of four rectangles *congruent* to $R(3a,a)$:

[picture of achieving this event... square annulus with four crossings in the four relevant rectangles...]

□

As a consequence, we obtain the following:

Theorem (Harris theorem)**.** *If $p > p_c$, then* with probability one (w.p.1) *the* dual *model does not contain an infinite cluster.*

Proof. This follows from two statements:

- For all $a > 0$, $C_p(a, a) \gtrsim \theta(p)^2$ (recall that $\theta(p) = \mathbb{P}(|C(0)| = \infty)$; here $\gtrsim$ means greater than or equal to up to a *numerical* constant).
- If $\liminf_a C_p(a, a) > 0$, then there is no dual infinite cluster.

The first item follows from the FKG inequality: Let $\{l_a, r_a, t_a, b_a\}$ denote the events that the origin is connected to the *left, right, top, bottom* boundaries, respectively. Then

- the four events are *increasing* and have the same probability by *symmetry*;
- it is clear that

$$l_a \cup r_a \cup t_a \cup b_a \supseteq \{0 \rightsquigarrow \infty\}$$

  so that together with the previous item

$$4 \cdot \mathbb{P}(l_a) \geq \mathbb{P}(l_a \cup r_a \cup t_a \cup b_a) \geq \theta(p);$$

- $l_a \cap r_a \subseteq \mathcal{C}(a, a)$.

[picture of origin being connected to boundary of square...]

We have therefore by the *FKG inequality* that

$$C(a,a) \geq \mathbb{P}(l_a \cap r_a) \geq \mathbb{P}(l_a) \cdot \mathbb{P}(r_a) = \mathbb{P}(l_a)^2 \geq \left(\frac{1}{4} \cdot \theta(p)\right)^2.$$

For the second item, $\liminf_a C(a,a) > 0$ means that $A_p(a, 3a)$ is *uniformly bounded below* as $a \to \infty$. So given any $N \in \mathbb{N}^+$, we can set up N *concentric logarithmic annuli*:

$$A(a, 3a), A(3a, 9a), A(9a, 27a), ...A((N-1)a, 3(N-1)a),$$

where $A_p(k \cdot a, 3k \cdot a) > \alpha > 0$, for some α. Also, the presence or absence of circuits in these annuli are clearly *independent* events.

[picture of dual origin being severed from ∞ by sequence of annuli...]

Therefore,

$$\mathbb{P}(|\mathcal{C}(0^*)| = \infty) \leq \mathbb{P}\left(\bigcap_{k=1}^{N} A^c(ka, 3ka)\right) \leq (1-\alpha)^N$$

for any N and so there is no dual infinite cluster. □

Exercise. Redo the proof of the first item in the above theorem using the *square root trick* to show that

$$C(a,a) \geq (1 - (1-\theta(p))^{1/4})^2.$$

Rescaling. More dramatic results can be derived by applying the RSW estimates on many scales. We start with the following *rescaling lemma*:

Lemma. *Suppose at some scale $a > 0$, we have $C(2a, a) \geq 1 - c\lambda$ with $c = 1/16$ for some $\lambda > 0$, then*

$$C(2^{k+1}a, 2^k a) \geq 1 - c\lambda^{2^k}, \quad \forall k \geq 1.$$

Proof. First we note that

[picture crossing of $R(4a, a)$ as "stitching" of 3 crossings of $R(2a, a)$ and 2 crossings of $R(a, a)$...]

$$C(4a, a) \geq C(2a, a)^3 \cdot C(a, a)^2 \geq C(2a, a)^4,$$

where the last inequality is since $C(2a, a) \leq C(a, a)^2$:

[picture of crossing of $R(2a, a)$, with dash in middle, implying crossings of the two squares...]

Therefore,

$$\begin{aligned} C(4a, a) &\geq (1 - c\lambda)^4 \quad (= 1 - 4c\lambda + O(\lambda^2) \quad) \\ &\geq 1 - 4c\lambda. \end{aligned}$$

Next we double in the *vertical* direction. Restricting crossing to the top or bottom half (and these events are *independent*, we have that

$$\begin{aligned} C(4a, 2a) &\geq 1 - \mathbb{P}(\{\text{no crossing in top half}\} \cap \{\text{no crossing in bottom half}\}) \\ &= 1 - (1 - C(4a, a)) \cdot (1 - C(4a, a)) \\ &\geq 1 - (4c\lambda)^2 \qquad\qquad (\text{since } 1 - C(4a, a) \leq 4c\lambda) \\ &= 1 - c\lambda^2, \end{aligned}$$

since $c = 1/16$.

[picture of crossing of double rectangle by *restricting* to each half... versus picture of crossing which traverses *both* halves...]

Iteration now gives the result: Assume as *inductive hypothesis* that $R(2^k a, 2^{k-1} a) \geq 1 - c\lambda^{2^{k-1}}$ and run the above argument to double the scale both horizontally and vertically. $\square$

As a consequence we can characterize *supercriticality* by *crossing probabilities* of squares:

Theorem.

$$p > p_c \quad \Longleftrightarrow \quad \liminf_{a\to\infty} C(a,a) = 1.$$

Proof. First suppose $\liminf_{a\to\infty} C(a,a) = 1$. Let

$$0 < \lambda < 1$$

and define the *length scale* $0 < L_0 < \infty$ to be large enough so that the *rescaling hypothesis* of the previous lemma is satisfied:

$$L_0(p,\lambda) = \inf_a \{a : C(2a,a) > 1 - c\lambda\}, \qquad c = 1/16.$$

We know that $L_0 < \infty$ since recall that by the RSW estimates

$$C(2a,a) \geq C(\frac{3}{2} \cdot a, a)^2 \cdot C(a,a) \geq (1 - \sqrt{1 - C(a,a)})^3 \cdot C(a,a),$$

and the right hand side tends to 1 by assumption.

We can now perform an *overlapping rectangles* construction to create a connection to ∞:

[picture of connection to infinity via crossings of larger and larger overlapping rectangles of the same aspect ratio... alternately "horizontal" and "vertical"...]

The probability of all such connections being realized is $\prod_{k\geq 1}(1-c\lambda^{2^k}) > 0$ (since $\lambda < 1$); given these connections, the origin is connected to infinity if we occupy L_0 vertical bonds along the x–axis. Thus,

$$\mathbb{P}_p(\{0 \rightsquigarrow \infty\}) \geq p^{L_0} \cdot \prod_{k\geq 1}(1 - c\lambda^{2^k}) > 0.$$

Conversely suppose $p > p_c$. By the rescaling lemma it is sufficient to show that

$$\exists a > 0, \quad C(2a, a) > 1 - c\lambda \quad \text{for some } \lambda < 1, c = 1/16.$$

We have that $(R_n := R(n, n))$

$$\begin{aligned}\mathbb{P}_p(\{(\partial R_n \rightsquigarrow \infty\}) &\geq \mathbb{P}_p(\{\exists x \in R_n : x \rightsquigarrow \infty\}) \\ &\underset{n\,\to\,\infty}{\longrightarrow} \mathbb{P}_p(\{\exists \text{ an infinity cluster}\}) \\ &= \psi(p) = 1.\end{aligned}$$

(Recall that $\psi(p) = 1$ follows from $\theta(p) > 0$ via the Kolmogorov 0–1 law.) Thus given any $\varepsilon > 0$, for $n \geq n_0$ sufficiently large,

$$\mathbb{P}_p(\{\partial R_n \rightsquigarrow \infty\}) > 1 - \varepsilon.$$

Next let $L \gg n$ and consider "*coarse grained*" events $\{l_{n,L}, r_{n,L}, t_{n,L}, b_{n,L}\}$ that there is a connection from ∂R_n to the left, right, top, bottom of ∂R_L.

[picture of coarse grain connection between small box and large...]

As before, we have

$$\varepsilon > 1 - \mathbb{P}_p(\{\partial R_n \rightsquigarrow \infty\}) = \mathbb{P}_p(\{\partial R_n \rightsquigarrow \infty\}^c) \geq \mathbb{P}_p(l^c_{n,L} \cap r^c_{n,L} \cap t^c_{n,L} \cap b^c_{n,L}).$$

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Since all these are *decreasing events*, we have by the FKG inequality that

$$\mathbb{P}_p(l_{n,L}^c) \leq \varepsilon^{1/4},$$

and similarly for the others. Again by the FKG inequality,

$$\mathbb{P}_p(l_{n,L} \cap r_{n,L}) \geq (1-\varepsilon^{1/4})^2.$$

Now a *left right* crossing of r_L emerges if there is a blue circuit in $A_{n,L}$ (an event which is *independent* of $l_{n,L}, r_{n,L}$, etc.).

Let us now take L *large* and set up logarithmically many annuli, each of which gives an *independent* chance of containing a *blue circuit*, all with probability bounded by some $\alpha > 0$: Indeed, we have that $C(a,a) \geq \left(\frac{1}{4}\cdot\theta(p)\right)^2 > 0$ on *all scales* a (since $p > p_c$) as in the proof of the Harris theorem and so by RSW, the probability of existence of a *circuit on all scales* is also *strictly bounded from below* (thanks for Helge Krüger for pointing out that some argument along these lines was missing in the version given in lecture).

[picture of construction of left right crossing from "half coarse grain crossings" and a circuit...]

This finally yields by the FKG inequality the estimate

$$\begin{aligned} C(L,L) &\geq \mathbb{P}_p\left((\ell_{n,L}\cap r_{n,L}) \ \cap \ \Big(\bigcup_{O(\log(L/n)) \ \text{annuli}} \{\exists \text{ blue circuit}\}\Big)\right) \\ &\geq \mathbb{P}_p(\ell_{n,L}\cap r_{n,L}) \cdot \mathbb{P}_p\left(\bigcup_{O(\log(L/n)) \ \text{annuli}} \{\exists \text{ blue circuit}\}\right) \\ &\gtrsim (1-\varepsilon^{1/4})^2 \cdot \log(L/n)\alpha, \end{aligned}$$

which can clearly be made arbitrarily close to one. $\square$

Exercise. Show that if $p > p_c$, then for all $a > 0$,

$$C(a, a) \geq \theta(p).$$

(Hint: consider a "9–square" construction and its shift and use translation invariance to show that any connection $0 \rightsquigarrow \infty$ must cross a square of any scale.)

We can now also show *continuity of transition:*

Theorem. *The function $\theta(p)$ is continuous at p_c. In particular $\theta(p_c) = 0$.*

Proof. First we show that $\theta(p)$ is continuous from the right. To this end note that

- $C_p(n, n)$ is a *continuous increasing* function of p and
- $C_p(n, n) \searrow \theta(p)$.

Now right continuity follows since in general if $f_n \searrow g$ with f_n *continuous increasing*, then g is *increasing* and *right continuous* (choose n such that $f_n(x) - f(x) < \varepsilon$; then choose δ such that $f_n(x + \delta) - f_n(x) < \varepsilon$).

For continuity from the left it suffices to prove $\theta(p_c) = 0$. Suppose $\theta(p_c) > 0$. Then by the previous theorem, we can find L_0 such that

$$C_{p_c}(L_0, L_0) > 1 - c\lambda$$

as in the hypothesis of the scaling lemma. But since $C_p(L_0, L_0)$ is a *polynomial in p* it is *continuous*, so for ε sufficiently small

$$C_{p_c - \varepsilon}(L_0, L_0) > 1 - c\lambda,$$

but then by the previous theorem $\theta(p_c - \varepsilon) > 0$, a contradiction. $\square$

Symmetry and Scale Invariance. Let us now observe that overall symmetry (or balance between the two colors) implies scale invariance of "all" crossing events.

Corollary (Hexagonal tiling at criticality)**.** Consider hexagonal tiling at $p = 1/2$ as before. Then for any $a, b \in \mathbb{N}^+$ and $k \in \mathbb{N}^+$,

$$C(2^k a, b) \geq 16^{-k} > 0.$$

Proof. Here by blatant symmetry given any "perfect square" *at any scale*, the probability of a crossing is 1/2. Thus,

$$C(2^k a, a) \geq \frac{1}{4} C(2^{k-1} a, b)^2 \geq \cdots \geq \left(\frac{1}{4}\right)^k \cdot C(a, a)^{2k} = \left(\frac{1}{4}\right)^{2k}.$$

□

Here, of course, the actual bound is not so important as the fact that the crossing probability of ever longer rectangles remain *strictly bounded from below*. *Scale invariance* follows from overall symmetry of the (*critical*) model (the color of each hexagon can be switched for free); once the probability of crossing of a simple shape (e.g., square) is seen to be scale invariant, then by RSW estimates probabilities of crossings of more complicated objects are also scale invariant.

References.

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