II. Basic Techniques

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Monotonicity and the FKG Inequality. Let $m < \infty$ and $\Omega_m = (\{0,1\}^m, \preceq)$.

- This corresponds to percolation on a *finite* lattice.
- We are equipping elements of Ω_m with the usual partial order:

$$(\omega_1, \ldots, \omega_m) \preceq (\omega'_1, \ldots, \omega'_m) \quad \Longleftrightarrow \quad \omega_1 \leq \omega'_1, \ldots, \omega_m \leq \omega'_m.$$

- This partial order immediately induces a notion of *monotonicity* for *functions.* Suppose $f : \Omega_m \to \mathbb{R}$, then we say f is *increasing* if

$$f(\omega) \leq f(\omega') \quad \text{whenever} \quad \omega \preceq \omega'.$$

- Equivalently, an *event* $A \subset \Omega_m$ is increasing if its *indicator function* $\mathbb{I}_A$ is increasing.

Definition. Let μ be a probability measure on Ω_m. Then μ has *positive correlations* (or the *FKG property*) if given two increasing functions f and g,

$$\mathbb{E}_\mu(fg) \geq \mathbb{E}_\mu(f) \cdot \mathbb{E}_\mu(g).$$

Equivalently, if A and B are two increasing events, then

$$\mu(A \cap B) \geq \mu(A) \cdot \mu(B).$$

Theorem (FKG inequality)**.** *Let $p_1, \ldots, p_m \in [0,1]$ and let μ be product measure on Ω_m:*

$$\mu = \prod_{k \in [m]} \mu_k; \qquad \mu_k(\omega_k = 1) = p_k, \quad \mu_k(\omega_k = 0) = 1 - p_k, \quad k = 1, \ldots, m.$$

Then μ has positive correlations.

Proof. This is more apparent if we rewrite the required statement as $\mu(A \mid B) \geq \mu(A)$: Since both events are increasing, the chances of A happening *conditioned* on B already having happened increases.

We proceed by induction on m. For $m = 1$ we have

$$\begin{aligned}
\mathbb{E}(fg) - \mathbb{E}(f) \cdot \mathbb{E}(g) &= p \cdot f_1 g_1 + (1-p) \cdot f_0 g_0 - [p \cdot f_1 + (1-p) \cdot f_0] \cdot [p \cdot g_1 + (1-p) \cdot g_0] \\
&= (p - p^2) \cdot f_1 g_1 + (1 - p - (1-p)^2) \cdot f_0 g_0 - p(1-p) \cdot (f_1 g_0 + f_0 g_1) \\
&= (p - p^2) \cdot (f_1 g_1 + f_0 g_0 - f_1 g_0 - g_1 f_0) \\
&= p(1-p) \cdot ((f_1 - f_0)(g_1 - g_0)) \\
&\geq 0,
\end{aligned}$$

since f, g are both increasing.

For general m we use conditioning: By independence, we have

$$\mu(\cdot \mid \omega_m) = \mu_1 \times \cdots \times \mu_{m-1} := \mu_{\mathbf{m-1}},$$

i.e., the *conditional measure* on the last coordinate is the product measure on the first $m-1$, so by the *induction hypothesis*,

$$\mathbb{E}_{\mathbf{m-1}}(fg \mid \omega_m) \geq \mathbb{E}_{\mathbf{m-1}}(f \mid \omega_m) \cdot \mathbb{E}_{\mathbf{m-1}}(g \mid \omega_m),$$

pointwise for each ω_m.

Also, since f is increasing, it is the case that

$$\mathbb{E}_{\mathbf{m-1}}(f(\cdot\ ,\ \omega_m = 1)) \geq \mathbb{E}_{\mathbf{m-1}}(f(\cdot\ ,\ \omega_m = 0)),$$

so the *functions*

$$\mathbb{E}_{\mathbf{m-1}}(f \mid \omega_m) \quad \text{and} \quad \mathbb{E}_{\mathbf{m-1}}(g \mid \omega_m)$$

are increasing in the last coordinate so the statement for $m = 1$ can be applied to obtain

$$\mathbb{E}_{\mu_m}\left[\mathbb{E}_{\mathbf{m-1}}(f \mid \omega_m) \cdot \mathbb{E}_{\mathbf{m-1}}(g \mid \omega_m)\right] \geq \mathbb{E}_{\mu_m}\left[\mathbb{E}_{\mathbf{m-1}}(f \mid \omega_m)\right] \cdot \mathbb{E}_{\mu_m}\left[\mathbb{E}_{\mathbf{m-1}}(g \mid \omega_m)\right].$$

Concatenating the above, we have

$$\begin{aligned}\mathbb{E}(fg) &= \mathbb{E}_{\mu_m}[\mathbb{E}_{\mathbf{m-1}}(fg \mid \omega_m)] \\ &\geq \mathbb{E}_{\mu_m}\left[\mathbb{E}_{\mathbf{m-1}}(f \mid \omega_m) \cdot \mathbb{E}_{\mathbf{m-1}}(g \mid \omega_m)\right] \\ &\geq \mathbb{E}_{\mu_m}\left[\mathbb{E}_{\mathbf{m-1}}(f \mid \omega_m)\right] \cdot \mathbb{E}_{\mu_m}\left[\mathbb{E}_{\mathbf{m-1}}(g \mid \omega_m)\right] \\ &= \mathbb{E}(f) \cdot \mathbb{E}(g).\end{aligned}$$

□

Example.

[picture of crossing of square from x to y *and* from z to w...]

We can remove the restriction to finite volume:

Corollary. Let (Ω, μ) be a probability space where μ is a product measure (in particular $\mu = \prod_{e \in \mathbb{E}} \mu_e$, where each $p_e \equiv p$ for some $0 \leq p \leq 1$) and X, Y are random variables such that

$$\mathbb{E}(X^2) < \infty, \quad \mathbb{E}(y^2) < \infty,$$

Then

$$\mathbb{E}(XY) \geq \mathbb{E}(X) \cdot \mathbb{E}(Y).$$

Proof. We condition on the state of a finite number of edges and define

$$X_n := \mathbb{E}(X \mid \omega_1, \ldots, \omega_n), \quad Y_n := \mathbb{E}(X \mid \omega_1, \ldots, \omega_n).$$

The *martingale convergence theorem* implies that $X_n \to X, Y_n \to Y$ a.s. and in L^2.

Also, by Cauchy–Schwarz (here $\|X\| = (\mathbb{E}X^2)^{1/2}$)

$$\mathbb{E}|X_n Y_n - XY| \leq \|X_n - X\| \cdot \|Y_n\| + \|X\| \cdot \|Y_n - Y\| \to 0.$$

The conclusion now follows from the finite volume case which gives for all n

$$\mathbb{E}(X_n Y_n) \geq \mathbb{E}(X_n) \cdot \mathbb{E}(Y_n).$$

$\square$

Exercise.

1. Show that the FKG inequality implies that if A is an *increasing* event and B is a *decreasing* event then
$$\mathbb{P}(A \cap B) \leq \mathbb{P}(A) \cdot \mathbb{P}(B).$$

2. Let μ and ν be measures on Ω_m. We say that μ *stochastically dominates* ν (written $\mu \succeq \nu$) if for every *increasing* function
$$\mathbb{E}_\mu(f) \geq \mathbb{E}_\nu(f).$$
Show that given a measure ν with positive correlations and an *increasing* function F, the measure
$$\mu = \frac{F}{\mathbb{E}_\nu(F)} \cdot \nu$$
stochastically dominates ν.

3. Let μ_p denote the product measure generated by percolation on $\mathbb{Z}$. Conclude from the previous item that if $p_1 > p_2$, then $\mu_{p_1} \succeq \mu_{p_2}$, so that in particular if A is an *increasing* event, then
$$\mathbb{P}_{p_1}(A) \geq \mathbb{P}_{p_2}(A).$$

4. (Strassen's theorem) Let $(\Omega, \preceq)$ be a (discrete) product probability space with a partial ordering and let μ and ν be measures on Ω. A necessary and sufficient condition for $\mu \succeq \nu$ is that there exists a *coupling* $M(\omega, \eta)$ with *marginals* μ and ν (i.e.,
$$\sum_\eta M(\omega, \eta) = \mu(\omega), \quad \sum_\omega M(\omega, \eta) = \nu(\eta)$$

)

such that

$$\mathbb{P}_M(\{(\omega,\eta) : \omega \succeq \eta\}) = 1.$$

Disjoint Occurrence and the BKR Inequality. The "reverse inequality" to the FKG inequality involves the notion of *disjoint occurrence*.

Definition. Suppose A is an event which only depends on a *finite* number of edges $e_1, \ldots, e_m$, let $S \subset [m]$ and let ω be a configuration. Define the *cylinder set* generated by (I_A, ω) by

$$\mathcal{C}(S,\omega) = \{\omega' : \omega' \mid_S = \omega\}.$$

- We say that A *occurs* on (S,ω) if

$$\mathcal{C}(S,\omega) \subseteq A,$$

- Now let A and B be two such events. Then A and B *occur disjointly on* ω if there exists *disjoint* subsets of $[m]$ on which they occur. The set of all such configurations is denoted $A \circ B$, so that

$$(A \cap B \supseteq)\ A \circ B = \{\omega : \exists S, T \subset [m], \quad S \cap T = \emptyset, \quad \mathcal{C}(S,\omega) \subseteq A \text{ and } \mathcal{C}(T,\omega) \subseteq B\}.$$

Here we are allowing for the possibility that the events may not be increasing so it may well be the case that it is the *absence* of a particular edge which determines an event (corresponding to $\omega_e = 0$). In the simpler setting of increasing functions, it is sufficient to realize the condition as $\cup_{s\in S}\, e_s \subseteq A$.

We will prove the inequality for *increasing functions*:

Theorem. *Let A and B be increasing events depending on only* finitely *many edges. Then*

$$\mathbb{P}(A \circ B) \leq \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Proof. The idea is to imagine *splitting* all edges of relevance into two one by one and requiring A and B to avoid the "other" copy so that the probability of *disjoint* occurrence can only *increase* (or, the set $A \circ B$ in the larger space grows). When all this is done we will have created two copies of the model, corresponding to $\mathbb{P}(A) \cdot \mathbb{P}(B)$.

We will do this one edge at a time. First, given m, k we will look at the product space

$$\Omega_{m+k} = \{0,1\}^m \times \{0,1\}^k.$$

Given $k \geq 0$, we will consider two labels for configurations

$$\begin{aligned} \omega &= (\omega_1, \omega_2, \ldots, \omega_k, \quad \omega_{k+1}, \ldots, \omega_m) \\ \tau &= (\eta_1, \eta_2, \ldots, \eta_k, \quad \omega_{k+1}, \ldots, \omega_m). \end{aligned}$$

Next we define augmented events in Ω_{m+k}

$$\begin{aligned} A' &= \{(\omega, \eta) : \omega \in A\} \\ B'_k &= \{(\omega, \eta) : \tau \in B\} \end{aligned}.$$

(so the event B is required to utilize k of the η's) It is clear that

$$\begin{aligned} \mathbb{P}(A' \circ B'_0) &= \mathbb{P}(A \circ B) \\ \mathbb{P}(A \circ B_m) &= \mathbb{P}(A) \cdot \mathbb{P}(B). \end{aligned}$$

By induction (iteration) it is sufficient to show that

$$\mathbb{P}(A' \circ B'_{k-1}) \leq \mathbb{P}(A \circ B_k).$$

We first look at $A' \circ B'_{k-1}$ and focus on the state of the k^{th} edge ω_k so let us factor this out and write

$$\Omega_{m+(k-1)} =: \Delta \times \{0,1\}.$$

Now we decompose events in $A' \circ B'_{k-1}$ according to the utility of e_k:

$$\Delta \supseteq E_{k-1} := \{\delta \in \Delta : (\delta, 0) \in A' \circ B'_{k-1}\}$$
$$\Delta \supseteq D_{k-1} := \{\delta \in \Delta : (\delta, 0) \notin A' \circ B'_{k-1}, \quad (\delta, 1) \in A' \circ B'_{k-1}\}.$$

It is clear that $E \cap D = \emptyset$ and since the events are *increasing* (the events $A' \circ B'_\ell$ are still increasing so if $\omega_k = 0$ is sufficient for the occurrence of the event so would $\omega_k = 1$ be)

$$\mathbb{P}(A' \circ B'_{k-1}) = \mathbb{P}(E) + p \cdot P(D).$$

The set D_{k-1} can be further decomposed according to whether the index k is necessary to ensure the occurrence of B'_{k-1} in $A' \circ B'_{k-1}$:

$$D^\alpha_{k-1} = \{\delta \in D_{k-1} : \exists S, \quad k \in S, \quad \mathcal{C}(S, \omega) \in A\}$$
$$D^\beta_{k-1} = D \setminus D^a_{k-1},$$

so that in D^β_{k-1}, the presence of e_k is necessary for the occurrence of B'_{k-1} in $A' \circ B'_{k-1}$.

Now the space of relevance for $A' \circ B'_k$ is Ω_{m+k}, i.e., an additional *coordinate* η_k will be added. To conclude we observe that there is a natural *probability increasing injection*

$$A' \circ B'_{k-1} \hookrightarrow A' \circ B'_k,$$

such that

$$E_{k-1} \hookrightarrow E_{k-1} \times \{0, 1\}$$
$$D^\alpha_{k-1} \hookrightarrow D^\alpha_{k-1} \times \{0, 1\},$$

that is, since in the sets E_{k-1}, D^α_{k-1}, the k^{th} index is *not necessary* for the realization of B'_{k-1}, the addition of η_k can only *improve* the probability; on the other hand, for D^β_{k-1}, it is sufficient to interchange the role of η_k and ω_k (since there the k^{th} index is not necessary for the realization of A):

$$D^{\beta}_{k-1} \longrightarrow A' \circ B'_k :$$

$$(\omega_1, \dots, \omega_{k-1}, \quad \omega_k, \quad \omega_{k+1}, \dots, \omega_m, \eta_1, \dots, \eta_{k-2}, \quad \eta_{k-1}) \ \ (\in \Omega_{m+k-1})$$
$$\rightsquigarrow (\omega_1, \dots, \omega_{k-1}, \quad \eta_k, \quad \omega_{k+1}, \dots, \omega_m, \eta_1, \dots, \eta_{k-2}, \quad \eta_k, \quad \omega_k) \ \ (\in \Omega_{m+k}).$$

(Recall that by definition of B'_k it is then $(\eta_1, \dots, \eta_{k-1}, \omega_k)$ which fulfills B.) Finally it is clear that the image of this final map still contributes $p \cdot D^{\beta}_{k-1}$ to $\mathbb{P}(A' \circ B'_k)$. □

A more general version of the statement holds and has been established by Reimer:

Theorem (Reimer's inequality)**.** *Let $n \in \mathbb{N}$ and let $(\Omega_n, \mathcal{F} := 2^{\Omega_n}, \mu)$ be a product probability space. Then for all $A, B \in \mathcal{F}$,*

$$\mu(A \circ B) \le \mu(A) \cdot \mu(B).$$

For proof of this and further discussions we refer to Reference 3.

Example.

[picture of k disjoint crossings of an annuli...]

Exercise. Show that the BKR inequality implies the FKG inequality.

Pivotal Bonds and Russo's Formula. *Russo's formula* is a derivative formula for how the probability of an event changes with the parameter p.

Definition. Let A be an *increasing* event and let $\omega \subset \mathbb{E}$ be a configuration. An edge $e \in \mathbb{E}$ is an *articulation bond* (*pivotal bond*) if

$$\mathbb{I}_A = \begin{cases} 1 & \text{if } \ e \in \omega \\ 0 & \text{if } \ e \notin \omega, \end{cases}$$

that is, *given* $\omega \setminus \{e\}$, the occurrence or not of A is determined by the state of e. We denote the (random) set of pivotal bonds by $\delta A(\omega)$.

Theorem (Russo's formula)**.** *Let A be an increasing event depending on* finitely *many edges, i.e., there exists some $0 \le \ell < \infty$ such that $A \subset \sigma(e_1, \ldots, e_\ell)$. Then*

$$\frac{d}{dp}\mathbb{P}_p(A) = \mathbb{E}_p(|\delta A|).$$

Proof. Let

$$\vec{p} = (p_1, \ldots, p_\ell)$$

be *dummy variables* corresponding to parameters at the edges $e_1, \ldots, e_\ell$. Then since these edges determine A (so A is independent of the states of all other edges and their corresponding parameters) we have by the *chain rule* that

$$\frac{d}{dp}\mathbb{P}_p(A) = \sum_{k=1}^{\ell} \frac{\partial}{\partial p_k}\mathbb{P}_p(A) \mid_{p_k = p} .$$

Next for each edge e_k we can write the *Bayesian decomposition*

$$\mathbb{P}_p(A) = \mathbb{P}_p(e_k \notin \delta A) \cdot \mathbb{P}_p(A \mid e_k \notin \delta A) + \mathbb{P}_p(e_k \in \delta A) \cdot \mathbb{P}_p(A \mid e_k \in \delta A).$$

Since whether or not e_k is pivotal is determined by $\omega \setminus \{e_k\}$, $\mathbb{P}_p(e_k \in \delta A)$ and $\mathbb{P}_p(e_k \notin \delta A)$ are independent of the state of e_k and hence p_k; also, since A is *increasing*, the same is true of $\mathbb{P}_p(A \mid e_k \notin \delta A)$. Therefore

$$\frac{\partial}{\partial p_k}\mathbb{P}_p(e_k \in \delta A) = 0, \quad \frac{\partial}{\partial p_k}\mathbb{P}_p(e_k \notin \delta A) = 0, \quad \text{and} \quad \frac{\partial}{\partial p_k}\mathbb{P}_p(A \mid e_k \notin \delta A) = 0.$$

On the other hand (by definition)

$$\mathbb{P}_p(A \mid e_k \in \delta A) = p_k.$$

Altogether we have

$$\begin{aligned}
\frac{d}{dp}\mathbb{P}_p(A) &= \sum_{k=1}^{\ell} \frac{\partial}{\partial p_k}\mathbb{P}_p(A) \mid_{p_k=p} \\
&= \sum_{k=1}^{\ell} \frac{\partial}{\partial p_k}[p_k \cdot \mathbb{P}_p(e_k \in \delta A)]_{p_k=p} \\
&= \sum_{k=1}^{\ell} \mathbb{P}_p(e_k \in \delta A) \ \ (= \sum_{k=1}^{\ell} \mathbb{E}_p(\mathbb{I}_{\{e_k \in \delta A\}}) = \mathbb{E}\left(\sum_{k=1}^{\ell} \mathbb{I}_{\{e_k \in \delta A\}}\right)) \\
&= \mathbb{E}_p(|\delta A|).
\end{aligned}$$

□

Example.

[picture of e being a pivotal edge for a left right crossing of a square: 4–arm picture...]

Exercise. Let $0 < p_1 \leq p_2 \leq 1$ and let A be an *increasing* event depending on m edges. Show that

$$\mathbb{P}_{p_2}(A) \leq (p_2/p_1)^m \cdot \mathbb{P}_{p_1}(A).$$

References.

1. Some lecture notes from a percolation course by L. Chayes, Spring '05, UCLA.

2. *Percolation* by G. Grimmett. Springer–Verlag, Berlin Heidelberg (1999).

3. *The Van Den Berg-Kesten-Reimer Inequality: A Review* by C. Borgs, J. T. Chayes, and D. Randall. *http://research.microsoft.com/apps/pubs/default.aspx?id=69686*