IV. Characteristic Length and $p_c = 1/2$

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Here we will show that percolation for the square lattice in $d = 2$ has $p_c = 1/2$. Similar arguments yield the same result for *hexagonal tiling* (or, equivalently, *site percolation on the triangular lattice*).

Correlations and Characteristic Length. We first introduce the *connectivity function*.

Definition. Let $x \in \mathbb{Z}^d$ and consider the *event*

$$T_{0x} = \{\omega : x \in C_\omega(0)\}$$

that x is connected to the origin.

[picture of $0 \rightsquigarrow x$...]

The *connectivity function* is the probability of this event:

$$\tau_{0x} := \mathbb{P}_p(T_{0x}).$$

The observation that the event $\{0 \rightsquigarrow x\} \supseteq \{0 \rightsquigarrow y\} \cap \{y \rightsquigarrow x\}$ implies *subadditivity* and hence the existence of a certain limit.

[picture $0 \rightsquigarrow x$ via $0 \rightsquigarrow y$...]

Proposition. Consider (without loss of generality) $x \in \mathbb{R}^d$ along the x–axis. Then the limit

$$m(p) := \lim_{x\to\infty}\left(-\frac{\log \tau_{0x}}{x}\right) = \inf_{x\geq 1}\left(-\frac{\log \tau_{0x}}{x}\right) \geq 0$$

exists and we have the *a priori* bound

$$\tau_{0x} \leq e^{-m(p)x},$$

so that in particular $m(p)$ is *decreasing* as a function of p.

Proof. Let $x, y \in \mathbb{Z}^d$, then since as observed $T_{0x} \supseteq T_{0y} \cap T_{yx}$, we have by the *FKG inequality* and *translation invariance* that

$$\tau_{0x} \geq \tau_{0y} \cdot \tau_{0(y-x)}.$$

It follows that

$$\log \tau_{0x} \geq \log \tau_{0y} + \log \tau_{0(y-x)},$$

so $(-\log \tau_{0x}) \geq 0$ (since $0 \leq \tau_{0x} \leq 1$ is a probability) is *subadditive*. The existence of limit now follows as in the case of the *connectivity constant* for SAW. The *a priori* estimate follows from the realization of $m(p)$ as an *infimum*. □

Next we observe that $m(p)$ is equivalent to the *length scale* $L_0^*(p)$ defined via the *rescaling hypothesis* for the *dual model*:

Proposition. Let us define $L_0^*(p,\lambda)$ to be the the *smallest* length for which the *dual model* satisfies the scaling hypothesis for $c = 1/16$ some $\lambda > 0$ (here we write C^* to emphasize we are describing crossing in the *dual model*)

$$C^*(2L_0^*, L_0^*) \geq 1 - c\lambda,$$

(so that from the *scaling lemma*

$$C^*(2^{k+1}L_0^*, 2^k L_0^*) \geq 1 - c\lambda^{2^k}. \qquad)$$

Then for *suitable choice of* λ, there exists constants c', c'' such that

$$\frac{1}{L_0^*} \leq m \leq \frac{c'}{L_0^*} + \frac{c'' \log L_0^*}{L_0^*}.$$

Proof. First it is observed that if the four (overlapping) $2L \times L$ rectangles around the origin are all crossed (the *long way*) by *dual* bonds, then $T_{0(L,0)}$ cannot occur:

[picture of 0 severed from $(L, 0)$ with $L, -L$ etc., labeled...]

Therefore by the *FKG inequality* applied to these four crossing events,

$$\tau_{0L} \leq 1 - C(2L, L)^4.$$

Setting $L = 2^k L_0^*$, we conclude from the *scaling lemma* that

$$\begin{aligned} \tau_{0L} &\leq 1 - (1 - c\lambda^{2^k})^4 \quad (\;\leq \frac{1}{4} \cdot \lambda^{2^k}) \\ &\leq e^{-\frac{1}{L_0^*} \cdot L} \qquad\qquad\quad (\;= e^{-2^k}), \end{aligned}$$

for suitable choice of λ. By the realization of $m(p)$ as the *infimum*, we immediately conclude

$$m(p) \geq \frac{1}{L_0^*}.$$

Conversely, we note that by *duality* the *absence* of a crossing in $R(L_0^* - 1, 2(L_0^* - 1))$ by the *dual model* is equivalent to a crossing the short way in the *original* model:

[picture of crossing long way of rectangle by dual and direct crossing in dash...]

This yields the estimate

$$1 - C(L_0^* - 1, 2(L_0^* - 1)) = \mathbb{P}_p(\bigcup_{a\in U, b\in V} T_{ab}) \leq \sum_{a\in U, b\in V} \tau_{ab},$$

where U, V denote the *sites* on the *long* edges of the rectangle. Noting that

- $\tau_{ab} \leq e^{-m(L_0^*-1)}, \quad \forall a \in U, b \in V;$
- $|U| \cdot |V| = 4(L_0^*)^2,$

we obtain the bound (since L_0^* is smallest such $C^*(2L_0^*, L_0^*) \geq 1 - c\lambda$)

$$c\lambda \leq 1 - C(L_0^* - 1, 2(L_0^* - 1)) \leq 4L_0^2 \cdot e^{-m(L_0-1)},$$

from which the bound $m \leq \frac{c'}{L_0^*} + \frac{c'' \log L_0^*}{L_0^*}$ follows by taking logarithms. □

Remark. Note that the above proposition also shows that up to constants and logarithms, the precise definition of m is not important (that is, exactly *how x tends to infinity* is not so essential).

Recall (from the *overlapping rectangles* construction) that

$$L_0(p) < \infty \quad \Longleftrightarrow \quad p > p_c,$$

so it must be the case that

$$L_0(p_c) = \infty.$$

Also, considering L_0^* to be associated to the *dual* model as in the previous proposition, we have from the above that

$$L_0(p^*) < \infty \quad \Longleftrightarrow \quad p^* > p_c^* \quad \Longrightarrow \quad p \leq p_c,$$

but this does *not* rule out the possibility that $L_0(p^*)$ becomes ∞ *strictly before* p_c (equivalently, $m(p)$ becomes 0 *strictly before* p_c). These considerations lead to the definition of the *susceptibility* and another *critical point*.

Susceptibility and Exponential Decay of Correlations. The *susceptibility* is defined as the *expected value* of $|\mathcal{C}(0)|$:

$$\begin{aligned}\chi(p) &:= \mathbb{E}_p(|\mathcal{C}(0)|(\omega)) \\ &= \mathbb{E}_p\left(\sum_{x\in\mathbb{Z}^d} \mathbf{1}_{T_{0x}}(\omega)\right) = \sum_{x\in\mathbb{R}^d} \tau_{0x}.\end{aligned}$$

Definition. The critical point π_c is then defined as

$$\pi_c = \sup\{p \in (0,1) : \ \chi(p) < \infty\}.$$

From this definition it is clear that

$$\pi_c \leq p_c.$$

Let us observe that since (roughly)

$$\begin{aligned}\mathbb{E}_p(|\mathcal{C}(0)|) &= \sum_{x\in\mathbb{R}^d} \tau_{0x} \\ &\lesssim \sum_{x\in\mathbb{R}^d} e^{-m(p)|x|} \\ &\sim \sum_{k} e^{-m(p)k} < \infty\end{aligned}$$

if $m(p) > 0$, it must be the case that $m(\pi_c) = 0$. In fact, $m(p)$ goes to zero *continuously*.

Proposition. There exists some $p' \leq \pi_c$ ($\leq p_c$) such that $\lim_{p\to p'} m(p) = 0$.

Proof. Let us consider *truncated* correlation functions:

$$\tau_{0x}^T = \mathbb{P}_p(\{0 \rightsquigarrow x \ \text{ inside } \{\vec{x} : -T \leq x_1, \ldots, x_d \leq T\}\}).$$

[picture of connection inside strip versus using bonds outside...]

It is clear that the associated $m^T(p)$ (so that, in particular, $\tau^T_{0x} \le e^{-m^T(p)x}$) is a *continuous, decreasing* function of p (continuous since τ^T_{0x} is a *polynomial* in p). It is also the case that $m^T(p) \searrow m(p)$ as $T \to \infty$:

- Since $\tau^T_{0n} \le \tau_{0n}$, it is clear that
$$m^T(p) \ge m(p).$$
- On the other hand, since $m(p)$ is realized as the *infimum*, given $\varepsilon > 0$,
$$\tau_{0n} \ge e^{-(m(p)-\varepsilon)n}, \qquad \forall n \ge n_0 \quad \text{sufficiently large.}$$
- Therefore, since $\tau^T_{0n} \searrow \tau_{0n}$, we have
$$\lim_{T\to\infty} e^{-m^T(p)n_0} \ge \lim_{T\to\infty} \tau^T_{0n} = \tau_{0n} \ge e^{-m(p-\varepsilon)n_0},$$
so we also have $\lim_{T\to\infty} m^T(p) \le m(p)$.

Since $m(p)$ is a *decreasing* limit of *continuous, decreasing* functions, it is *left* continuous.

Next we see that $m(p)$ is also *right* continuous: Suppose $m(p_0) > 0$. Then

$$L^*_0(p_0) \sim \frac{1}{m(p_0)} < \infty,$$

so that in the *dual* model, we have that

$$C^*(2L^*_0, L^*_0) \ge 1 - c\lambda.$$

Since $C^*(2L^*_0, L^*_0)$ is *continuous* in p, for $\varepsilon > 0$ sufficiently small, the same is true, that is

$$L^*_0(p) \le L^*_0(p+\varepsilon) < \infty \quad \Longrightarrow \quad m(p_0+\varepsilon) > 0.$$

Finally, if $m(\pi_c) > 0$, then the applying the *rescaling lemma* (to form *circuits* in ever larger annuli, via the *RSW estimates*) and the above *continuity argument* to the *dual* model we would deduce that $\chi(\pi_c+\varepsilon)<\infty$ (exercise) contradicting the definition of π_c.

[picture of dual circuit of scale L preventing connection to $|x|>L$...] $\square$

To complete the characterization of π_c as the point at which m becomes 0, we will need the converse to the above proposition ($p<\pi_c \Rightarrow m(p)>0$). This will be provided by the following *correlation inequality*:

Proposition (Lieb–Simon inequality)**.** Let D be a cube centered at the origin. For $z\in\partial D$, let

$$\tau'_{0z}=\mathbb{P}_p\{0 \rightsquigarrow z \ \text{ inside } D\}.$$

[picture of path contributing to τ'_{0z} together with τ_{0z}...]

Then for $x\notin D$,

$$\tau_{0x}\le \sum_{z\in\partial D}\tau'_{0z}\cdot\tau_{zx}.$$

(Note that the BK–inequality would immediately give

$$\tau_{0x}\le \sum_{z\in\partial D}\tau_{0z}\cdot\tau_{zx},$$

which is a worse bound than we have stated.)

Proof. This follows from the fact that

$$T_{0x} = \bigcup_{z \in \partial D} T'_{0z} \circ T_{zx},$$

where $\tau'_{0z} := \mathbb{P}_p(T'_{0z})$. This is understood as follows: let

$$C_D(0) = \mathcal{C}(0) \cap \bar{D}$$

be the cluster of the origin lying *entirely* inside $\bar{D}$. Then

$$T_{0x} = \{\omega : \exists z \in \partial D : \omega \in T'_{0z} \circ T_{zx}\}.$$

Indeed, given any $\omega \in T_{0x}$, *orient* a path (any path) $\gamma : 0 \rightsquigarrow x$, then

$$z = \{\gamma(t) : \gamma \text{ first exits } D \text{ at time } t\}.$$

Then clearly

$$\gamma([0,t]) \cap \mathbb{Z}^d \subset C_D(0).$$

The remainder of γ is either *outside* $C_D(0)$ or, if γ re–enters D and intersects $C_D(0)$ again at some point z', then we may *replace* the *first* part of γ by some path $\gamma' : 0 \rightsquigarrow z'$ lying entirely inside D and continue with γ until the next time γ exits ∂D.

[picture of connection between 0 and x with “rewiring” at z'...]

That this procedure terminates shows that $\omega \in T'_{0z} \cap T_{zx}$, since it produces a path $\tilde{\gamma} : 0 \rightsquigarrow x$ such that the portion of $\tilde{\gamma}$ from the origin to the first time it exits ∂D lies inside

$C_D(0)$ and the remainder lies outside C_D.

We can now finish by the *BK inequality*:

$$\tau_{0x} = \mathbb{P}_p\Big(\bigcup_{z\in\partial D} T'_{0z} \circ T_{zx}\Big) \leq \sum_{z\in\partial D} \mathbb{P}_p(T'_{0z} \circ T_{zx}))$$
$$\leq \sum_{z\in\partial D} \tau'_{0z} \cdot \tau_{zx}.$$

□

Theorem. *The* critical point π_c *characterized m:*

$$p < \pi_c \Longleftrightarrow m(p) > 0 \quad \textit{and} \quad \lim_{p\searrow\pi_c} m(p) = 0.$$

Proof. It only remains to prove that $p < \pi_c \Longrightarrow m(p) > 0$. We have that

$$\chi(p) = \sum_x \tau_{0x} < \infty \quad \Longrightarrow \quad e^{-\alpha} =: \sum_{z\in\partial D} \tau'_{0z} < 1, \quad \text{for } \|D\| := \operatorname{diam}(D) \text{ sufficiently large.}$$

For $x \gg \|D\|$, by the *Lieb–Simon inequality* we have

$$\tau_{0x} \leq e^{-\alpha} \cdot \sum_{z\in\partial D} e^{\alpha}\tau'_{0z} \cdot \tau_{zx} =: e^{-\alpha} \cdot \sum_{z\in\partial D} w_z \cdot \tau_{zx},$$

where it is noted that

$$\sum_{z\in\partial D} w_z = 1.$$

We may now apply the inequality to τ_{zx} to obtain

$$\tau_{0x} \leq e^{-2\alpha} \cdot \sum_{z\in\partial D} w_z \sum_{z'\in\partial(D+z)} w_{z'-z} \cdot \tau_{z'x} \leq e^{-2\alpha},$$

since $\tau_{z'x} \leq 1$ and the w's sum to 1.

[picture of one iteration, with $0, z, z', x$ labeled...]

Iterating this $|x|/\|D\|$ times by *translating* the relevant boxes and applying the inequality, we obtain that

$$\tau_{0x} \leq e^{-\alpha|x|/\|D\|} \cdot F(w,\tau) \leq e^{-\alpha|x|/\|D\|} \quad (\Longrightarrow \quad L_0^* < \infty \quad \Longrightarrow \quad m(p) > 0).$$

□

So far we have that if $p < \pi_c$, then there is *exponential decay of correlations* for the *direct model* and *finite characteristic length* $L_0^* \sim \frac{1}{m(p)}$ for the *dual model*, which, after applying the *rescaling lemma* implies that there is percolation in the dual model, i.e., $p^* > p_c^*$. Therefore, if the model is *self–dual* (the dual model is the same as the direct) *and we can show that*

$$p_c = \pi_c$$

then we would have that

$$\{ \;\; p_c = p_c^*, \quad p + p^* = 1, \quad p < p_c \Rightarrow p^* > p_c \;\; \} \quad \Longrightarrow \quad p_c = 1/2.$$

The Kesten Theorem. The goal here is to show that

$$[\pi_c, p_c] = \{p_c\},$$

that is, there is *no gap*. We already know that

- If $p \geq \pi_c$, then

  $$m = 0 \quad \Longrightarrow \quad L_0^* = \infty \quad \Longrightarrow \quad p^* \leq p_c^*,$$

  which implies that $\exists 0 < \sigma' < 1$ such that

  $$C^*(L,L) \leq \sigma', \quad \textit{at all scales } L,$$

  since otherwise the *rescaling lemma* can be applied to the *dual* model, *contradicting* $p^* \leq p_c^*$.

- If $p \leq p_c$, then $\exists 0 < \sigma < 1$ such that

$$C(L, L) \leq \sigma, \quad \textit{at all scales } L,$$

since otherwise the *rescaling lemma* applied in the *direct* model would yield *supercriticality*, contradicting $p \leq p_c$.

Therefore if $p \in [\pi_c, p_c]$, then the *crossing probability* is *severely constrained at all scales:*

$$1 - \sigma' \leq C(L, L) \leq \sigma, \quad \textit{at all scales } L.$$

We will use this and *Russo's formula* to deduce that if $p \in [\pi_c, p_c]$, then

$$\frac{d}{dp} C_p(L, L) \to \infty, \quad \text{as } L \to \infty,$$

so that in particular we can arrive at the contradiction that $\forall p \in [\pi_c, p_c]$ and $\forall \varepsilon > 0$ such that $p + \varepsilon \in [\pi_c, p_c]$,

$$\lim_{L \to \infty} C_{p+\varepsilon}(L, L) > 1,$$

and we are forced to conclude that $[\pi_c, p_c] = \{p_c\}$.

Let us first tally the relevant observations and ingredients:

- Russo's formula requires us to count the number of *articulation bonds* of the *crossing event* and thus our goal boils down to showing that the number of pivotal bonds tends to *infinity* as $L \to \infty$. It is easy to see that given ω, for an edge e to be an articulation bond for a *blue* left right crossing, the *dual* sites *above* and *below* (or to the *left* and *right*) of e must be connected to the *top* and *bottom* of the square:

[picture of horizontal and vertical articulation bond (two possibilities for vertical) with connection between the two halves of the blue crossing being disrupted by the dual connections to the top/bottom...]

- Next recall the notion of the *lowest* left right crossing and note that all *dual sites* below the lowest crossing must already be connected to the bottom of the square (otherwise, a lower crossing would be possible):

[picture of lowest crossing with all dual sites below connected to the bottom with the possibility of a lower crossing disrupted by such a connection...]

- Finally, the *scale invariant* estimates on the crossing probability for $p \in [\pi_c, p_c]$ implies that a careful *multiscale* construction and RSW estimates would lead to an estimate for the *number of articulation bonds* which *blows up* with L: we look at the lowest crossing restricted to the bottom half of the square, condition on the region formed by the "first" *articulation bond* and find many more in the unconditioned region.

[picture of "first" articulation bond and "wedge" unconditioned region formed and divided into scales...]

We start with some uniform estimates.

Lemma. *Let* $p \in [\pi_c, p_c]$ *and let* B_L *be the event of a* left right *crossing of* $R(L, L)$ *which takes place* entirely in the lower half of $R(L, L)$.

[picture B_L...]

Then there exists $0 < s < 1$ *such that for all* $p \in [\pi_c, p_c]$,

$$s \leq \mathbb{P}_p(B_L) \leq 1 - s, \quad \text{uniformly in } L.$$

(In particular, we may take

$$s = (1 - \sqrt{\sigma'})^3 \cdot (1 - \sigma'). \;)$$

Proof. This follows immediately from the bound for $R(L, L)$, since B_L is implied by a crossing of $R(L, \frac{1}{2}L)$ which can be bounded by $C_p(\frac{1}{2}L, \frac{1}{2}L)$ by the *RSW estimates*, so

$$\begin{aligned} 1 > \sigma > C(L, L) &\geq \mathbb{P}_p(B_L) \\ &\geq C_p(L, \frac{1}{2}L) \geq C_p(\frac{3}{4}L, \frac{1}{2}L) \cdot C_p(\frac{1}{2}L, \frac{1}{2}L) \\ &\geq \left(1 - \sqrt{1 - C_p(\frac{1}{2}L, \frac{1}{2}L)}\right)^3 \cdot C_p(\frac{1}{2}L, \frac{1}{2}L) \\ &\geq (1 - \sqrt{\sigma'})^3 \cdot (1 - \sigma') \\ &> 0 \end{aligned}$$

□

Lemma. *Let* $p \in [\pi_c, p_c]$ *and let*

$$\begin{aligned} Q_L = \{\omega : \quad &\exists \geq 1 \;\; \textit{articulation bond for } (L, L) \\ &\textit{in the } \text{bottom right } \textit{quadrant of } R(L, L)\}. \end{aligned}$$

[picture of Q_L: "four arm" centered at bottom right quadrant with left right crossing in the lower half...]

Then there exists $t(\sigma') > 0$ such that for all $p \in [\pi_c, p_c]$,

$$\mathbb{P}_p(Q_L \cap B_L) \geq t(\sigma'), \quad \text{uniformly in } L.$$

(In particular, we may take $t(\sigma') = s^2$ where s is from the previous lemma.)

Proof. Here we will make use of *conditioning* again. Let us enumerate the crossings of B_L:

$$B_L = \{\gamma_1, \ldots, \gamma_n\}$$

and let

$$\Pi_i = \{\omega : \ \gamma_i \text{ is the } \textit{lowest} \text{ crossing}\}, \qquad i = 1, \ldots, n,$$

so that $B_L = \cup_{i=1}^n \Pi_i$ as a *disjoint union*, so that

$$\begin{aligned} \mathbb{P}_p(\cdot \mid B_L) &= \frac{\mathbb{P}_p(\cdot \cap B_L)}{\mathbb{P}_p(B_L)} = \sum_i \frac{\mathbb{P}_p(\cdot \cap \Pi_i)}{\mathbb{P}_p(B_L)} \\ &= \sum_i \mathbb{P}_p(\cdot \mid \Pi_i) \cdot \frac{\mathbb{P}_p(\Pi_i)}{\mathbb{P}_p(B_L)} = \sum_i \mathbb{P}_p(\cdot \mid \Pi_i) \cdot \mathbb{P}_p(\Pi_i \mid B_L), \end{aligned}$$

where the last inequality is due to the tautology that $\mathbb{P}_p(\Pi_i) = \mathbb{P}_p(\Pi_i \cap B_L)$.

Certainly,

$$\mathbb{P}_p(Q_L) \geq \mathbb{P}_p(Q_L \cap B_L) = \mathbb{P}_p(Q_L \mid B_L) \cdot \mathbb{P}_p(B_L),$$

so we have from the previous *partitioning* that

$$\mathbb{P}_p(Q_L) \geq \mathbb{P}_p(B_L) \cdot \sum_i \mathbb{P}_p(Q_L \mid \Pi_i) \cdot \mathbb{P}_p(\Pi_i \mid B_L).$$

Now we use the observations that

- *conditionining on* Π_i means that there is a *dual* connection to the bottom "below" each site on Π_i so what is required to form an *articulation bond* is a *dual* connection to the top of $R(L, L)$;

- Π_i being the *lowest* crossing means that percolation in the *unconditioned* region above Π_i is *independent* of Π_i (that is, the conditioning here is basically trivial)

[picture of $R(L,L)$ divided into quadrants with the lowest crossing with conditioned region shaded and dual connection to the top in the correct quadrant...]

The above implies that

$$\begin{aligned}
\mathbb{P}_p(Q_L \mid \Pi_i) &\geq \mathbb{P}_p(\exists \text{ a } \textit{dual} \text{ crossing from the top of } R(L,L) \text{ to } \Pi_i \text{ in the } \textit{right} \text{ half of } R(L,L)) \\
&\geq \mathbb{P}_p(\exists \text{ a } \textit{dual top bottom} \text{ crossing of } R(L,L) \text{ in the } \textit{right} \text{ half of } R(L,L)) \\
&= \mathbb{P}_p(B_L) \\
&\geq s,
\end{aligned}$$

where $0 < s < 1$ is from the previous lemma.

Finally, altogether we therefore have that

$$\begin{aligned}
\mathbb{P}_p(Q_L) &\geq \mathbb{P}_p(B_L) \cdot \sum_i \mathbb{P}_p(Q_L \mid \Pi_i) \cdot \mathbb{P}_p(\Pi_i \mid B_L) \\
&\geq s^2 \cdot \sum_i \mathbb{P}_p(\Pi_i \mid B_L) \\
&= s^2 > 0.
\end{aligned}$$

□

Theorem (The Kesten theorem). *In two dimensions, $\pi_c = p_c$.*

Proof. To prove the theorem it remains to carry out the *conditioning on "wedge"* described earlier in order to estimate the *total* number of *articulation bonds*. A picture of the region of interest has already appeared, but let us note the important observations:

- First note that if Q_L occurs, then we may condition on the *lowest left right* crossing together with another *dual* path which is the *rightmost top bottom* crossing. Let us denote the resulting region containing the *top left* corner U (this is a *random* region).
- The restriction of the *articulation bond* to the *lower right* quadrant together with the event B_L implies that the *wedge* U contains the entire *top left* quadrant of $R(L, L)$.
- The region U is entirely *unconditioned*, namely, percolation in U is independent of the events Q_L, B_L.

We can now finish by performing *RSW estimates* in *annuli* on *many scales*: Let

$$\frac{1}{2}L = 3^N, \quad \text{some } N \in \mathbb{N}^+,$$

so that inside U there are (portions of) N *disjoint partial annuli*

$$a_1 \cap U, \ldots, a_N \cap U$$

each of which has *independent* probability of containing a *dual* circuit and clearly,

$$A_n^* \quad \Longrightarrow \quad \text{articulation bond at the terminal point of the circuit on } \Pi_i,$$

here A_n^* denotes the existence of a circuit in the *partial annuli* $a_n \cap U$.

[picture of multiscale construction with γ_i', τ_j labeled...]

Therefore, we may count the number of *articulation bonds* δ as follows:

$$\begin{aligned}\mathbb{E}_p(\delta) &\geq \mathbb{P}(Q_L \cap B_L) \cdot \mathbb{E}_p(\delta \mid Q_L \cap B_L) \\ &\geq t \cdot \sum_{i,j} \mathbb{E}_p(\delta \mid \gamma_i', \tau_j) \cdot \omega_{ij}.\end{aligned}$$

Here γ_i', τ_j denotes the two parts forming ∂U and

$$\begin{aligned}\omega_{ij} &= \mathbb{P}_p(\partial U = \gamma_i' \cup \tau_j) \\ &= \mathbb{P}_p(\{\gamma_i' \text{ is part of the } \textit{lowest left right crossing}\} \\ &\cap \{\tau_j \text{ is the } \textit{rightmost dual top bottom crossing}\}).\end{aligned}$$

By the *RSW construction* we have that

$$\begin{aligned}\mathbb{E}_p(\delta \mid \gamma_i', \tau_j) &\geq \sum_{n=1}^{N} A_n^* \\ &\geq N \cdot r \\ &= (\log L) \cdot r\end{aligned}$$

where

$$r := r(\sigma') \ (\geq C_p(3L, L)^4) \ > 0$$

is the uniform *lower* bound for A_n^* from the *RSW estimates* together with the estimates on $C_p(L, L)$ (we have bounded the probability of a *dual circuit* in the *partial annulus* $a_n \cap U$ by the probability of the existence of a circuit in the *full annulus* a_n). Therefore altogether,

$$\begin{aligned}\frac{d}{dp} C_p(L, L) &= \mathbb{E}_p(\delta) \\ &\geq (\log L) \cdot tr \cdot \sum_{ij} \omega_{ij}. \\ &= (\log L) \cdot tr\end{aligned}$$

Finally, given $\varepsilon > 0$ so that $p + \varepsilon \in [\pi_c, p_c]$, integration gives that

$$C_{p+\varepsilon}(L, L) \geq \varepsilon(\log L) \cdot tr > 1,$$

for L sufficiently large, since $tr = (tr)(\sigma) > 0$. □

Corollary. We have that $p_c = 1/2$ for *bond* percolation on the *square* lattice and *hexagonal tiling* (equivalently, *site* percolation on the *triangular* lattice).

Proof. Since both models are *self–dual* and satisfy the *RSW estimates*, the *BK* and *FKG* inequalities (and hence also the *rescaling lemma* and its consequences) apply, and this follows from the Kesten theorem and the discussion before this section.

(Here we mean *Whitney duality*: G^* is the dual of G if any *cycle* of G is a cut of G^*, and any *cut* of G is a cycle of G^*. Here a *cut* partitions the vertex set into two *disjoint* subsets, so in the context of percolation, if we draw a *blue cycle* and color everything else *yellow*, then we should have two disjoint clusters of *yellow*, each of which should be considered *connected*.) □

References

1. *Percolation and Random Media* by J. T. Chayes and L. Chayes. Lecture notes for Les Houches, summer 1984.

2. Warm thanks to attendees of these lectures for their questions and comments.