VIII. Kesten's Scaling Relations

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Here we use the previous results on *arm crossing events* to establish certain *scaling relations* relating various *exponents* for *macroscopic* quantities *near criticality.*

Various Quantities and Critical Exponents. Various *characteristic lengths*:

- (*Finite–size scaling*) This is the one defined via *crossing probabilities*: if $s', s > 0$ are such that $1 - s' \le C_{p_c}(L, L) \le s$ for all L, then for $0 < \varepsilon_0 < \min\{1 - s', s\}$,

$$L_{\varepsilon_0}(p) = \inf_n \{C_p(n, n) \le \varepsilon_0\}, \quad p < p_c;$$

$$L_{\varepsilon_0}(p) = \inf_n \{C_p^*(n, n) \le \varepsilon_0\}, \quad p > p_c.$$

- (*Mean radius of a finite cluster*) *Quadratic* mean of a *finite* cluster:

$$\xi(p) = \left(\frac{1}{\mathbb{E}_p(|\mathcal{C}(0)|; |\mathcal{C}(0)| < \infty)} \cdot \sum_x |x|^2 \cdot \mathbb{P}(0 \rightsquigarrow x, |\mathcal{C}(0)| < \infty) \right)^{1/2}$$

- (*Correlation length*) This controls the rate of *decay of correlations*:

$$\mathbb{P}(0 \rightsquigarrow x) \le e^{-m(p)\cdot|x|},$$

  where

$$\frac{1}{m(p)} = -\frac{\log \tau_n}{n},$$

  with

$$m(p) = \lim_{n\to\infty} \left(-\frac{\log \tau_n}{n} \right).$$

  Here

$$\tau_n := \sup_{x \in \partial R_n} \tau_{0,x} \ \ (= \sup_{x \in \partial R_n} \mathbb{P}(0 \rightsquigarrow x)\ ).$$

Notation. We list here some notation we will later use in order to reduce clutter:

- for $x \in \mathbb{Z}^2$, $|x| := |x|$;
- for $x \in \mathbb{Z}^2$, $T_{0x} := \{0 \rightsquigarrow x\}$;
- for $N \geq 0$, $\mathscr{F}(N) := \{\mathcal{C}(0) \subseteq R_N\}$, $\mathscr{F}(\infty) := \{|\mathcal{C}(0)| < \infty\}$.

Next we have various *critical exponents*: under the *assumption* that the relevant quantities behave as a *power law* in p *near criticality*, we *define* the exponents ν, β, γ as:

- (*Correlation length*)
$$m(p) \sim |p - p_c|^{-\nu}, \quad p \to p_c.$$
- (*Density of infinite cluster*)
$$\theta(p) \sim (p - p_c)^{\beta}, \quad p \searrow p_c.$$
- (*Average size of finite cluster*)
$$\chi(p) \sim |p - p_c|^{-\gamma}, \quad p \to p_c.$$

(Here $X \sim Y$ means there exists constants $c, c' > 0$ such that $X \leq cY, Y \leq c'X$.)

It turns out that these length scales are *all comparable* and consequently we will be able to relate *critical exponents* corresponding to various quantities.

Theorem. *Let ν, β, γ be as defined above and let $\alpha_1, \alpha_4 > 0$ be exponents for* one *and* four arm *crossing events at criticality:*

$$\mathbb{P}_{p_c}(A_{1,B}(0, n)) := \pi_1(n) \sim n^{-\alpha_1}, \quad \mathbb{P}(A_{4,BYBY}(0, n)) := \pi_4(n) \sim n^{-\alpha_4}.$$

Then ν, β, γ can be expressed in terms of α_1 and α_4:

$$\nu = 2 - \alpha_4, \quad \beta = \nu\alpha_1 = \alpha_1 \cdot (2 - \alpha_4), \quad \gamma = 2\nu \cdot (1 - \alpha_1) = 2(1 - \alpha_1) \cdot (2 - \alpha_4).$$

ν: Critical Exponent for the Characteristic Length(s). The result for ν is implied by the following:

Proposition. Let $s > 0$ denote the *scale invariant* bound on *crossing probabilities* at *criticality* as before. Then for any $0 < \varepsilon < \min\{1 - s', s\}$, we have

$$|p - p_c| \cdot (L_\varepsilon(p))^2 \cdot \pi_4(L_\varepsilon(p)) \sim 1.$$

Proof. For reason which should (hopefully) be clear by now, we will suppress the ε index in this proof. Let us assume *without loss of generality* that $p > p_c$. Then interpolating via $p(t) = (1-t)p_c + tp$ and using *Russo's formula* we have as before that

$$\begin{aligned}\mathcal{V}(p) := \mathcal{C}_p(L(p), L(p)) - \mathcal{C}_{p_c}(L(p), L(p)) &= \int_0^1 (p - p_c) \cdot \sum_{v \in R_{L(p)}} \mathbb{P}_t\left(v \rightsquigarrow_{4,\sigma_a} \partial R_{L(p)}\right) \\ &= (p - p_c) \cdot \sum_{v \in R_{L(p)}} \mathbb{P}_t\left(v \rightsquigarrow_{4,\sigma_a} \partial R_{L(p)}\right).\end{aligned}$$

(Note that no *absolute values* are necessary since by *monotonicity*,

$$p > p_c \quad \Longrightarrow \quad \mathcal{C}_p(L(p), L(p)) \geq \mathcal{C}_{p_c}(L(p), L(p)),$$

and we have a similar statement if it were the case that $p < p_c$.) The *left hand* side is of *order unity*, since by our *choice* of ε, we have that

$$\mathcal{V}(p) > s - \varepsilon > 0.$$

The statement would therefore follow if we show that

$$\sum_{v \in R_{L(p)}} \mathbb{P}_t\left(v \rightsquigarrow_{4,\sigma_a} \partial R_{L(p)}\right) \sim (L(p))^2 \cdot \pi_4(L(p)).$$

- Let $0 < \lambda < 1$ and consider the *shrunken box* $R_{(1-\lambda)L(p)}$. From our work before we already have that for all $v \in R_{(1-\lambda)L(p)}$,

$$\mathbb{P}_t\left(v \rightsquigarrow_{4,\sigma_a} \partial R_{L(p)}\right) \sim_\lambda \mathbb{P}_t\left(0 \rightsquigarrow_{4,\sigma_a} \partial R_{L(p)}\right),$$

and, also,

$$\mathbb{P}_t\left(0 \rightsquigarrow_{4,\sigma_a} \partial R_{L(p)}\right) \sim \mathbb{P}_{p_c}\left(0 \rightsquigarrow_{4,\sigma_a} \partial R_{L(p)}\right) = \pi_4(L(p)).$$

Therefore we already have a *lower bound*:

$$\sum_{v \in R_{L(p)}} \mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R_{L(p)}) \geq \sum_{v \in R_{L(p)}} \mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R_{(1-\lambda)L(p)}) \gtrsim (L(p))^2 \cdot \pi_4(L(p))$$
$$\implies \mathcal{V}(p) \gtrsim (p - p_c) \cdot (L(p))^2 \cdot \pi_4(L(p)).$$

- To finish it remains to show that contributions from $v \in R_{(1-\lambda)R_{L(p)}}$ contribute a *non–trivial* fraction to the sum. This entails an *extension of crossings* statement:

  Given any $\delta > 0$, there exists $\lambda_0 > 0$ such that *uniformly in* p' for p' between p and p^*, for any $n \leq L(p)$, $\lambda \leq \lambda_0$,

  $$|\mathcal{C}_{p'}((1-\lambda)n, n) - \mathcal{C}_{p'}(n,n)| \leq \delta.$$

  The crossing probability for the *smaller* box is clearly larger. The opposite inequality can be done by *RSW constructions:*

  - first we can assume the *left right crossing* does not go near the *upper right corner* with probability *in excess* of $1 - \lambda^\kappa$ for some $\kappa > 0$;
  - then *conditioning* on the *lowest* crossing, we may perform a RSW construction *above* the crossing to *continue it* to the *longer* box. *Conditioning* first on the existence of crossing in the *smaller* rectangle (and then the crossing being far from the corner, etc.) we conclude that

    $$\mathcal{C}_{p'}((1+\lambda)n, n) \geq \mathcal{C}_{p'}(n,n) \cdot (1 - \lambda^\kappa) \cdot (1 - \lambda^{\kappa'}) \geq (1-\delta) \cdot C_{p'}(n,n),$$

    for $\lambda > 0$ sufficiently small.

[picture square and slightly extended rectangle with upper right corner "sealed off" and RSW construction to "continue" crossing...]

- Therefore, given

$$0 < \delta \ll s - \varepsilon,$$

for $\lambda > 0$ appropriately chosen, we have that

$$\mathcal{C}_p((1-\lambda)L(p),(1-\lambda)L(p)) - 2\delta \leq \mathcal{C}_p(L(p),L(p)) \leq \mathcal{C}_p((1-\lambda)L(p),(1-\lambda)L(p)) + 2\delta,$$

which means that (much like the *variation of parameters* arguments before) the *difference* in crossing probabilities of $R_{L(p)}$ at *parameters* p *and* p_c can be estimated by changing the *parameter only* within the *smaller* square $R_{\lambda L(p)}$, up to an *error* $O(\delta)$ and so indeed,

$$\mathcal{V}(p) - O(\delta) \sim (p - p_c) \cdot \sum_{v \in R_{(1-\lambda)L(p)}} \mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R_{L(p)}).$$

[picture square and smaller square with boundary region shaded (parameter frozen)...]

□

Remark. The *continuation of crossing* argument used above should be *thought of* as implying a very simple version of a *continuity in domain* of crossing probability result.

As a corollary, we have:

Corollary. For any $0 < \varepsilon, \eta < \min\{1 - s', s\}$,

$$L_\varepsilon(p) \sim L_\eta(p).$$

So, in particular, $L(p)$ is indeed (up to a constant) *well–defined.*

Proof. Suppose $L_\varepsilon(p) \geq L_\eta(p)$. We have from the proposition that

$$(L_\varepsilon(p))^2 \cdot \pi_4(L_\varepsilon(p)) \sim (L_\eta(p))^2 \cdot \pi_4(L_\eta(p)),$$

since both are $\sim \frac{1}{|p-p_c|}$. Assume *without loss of generality* that $L_\varepsilon(p) \geq L_\eta(p)$, then by *quasi–multiplicativity* and using the (*existential*) exponent for π_4:

$$\begin{aligned}(L_\varepsilon(p))^2 \sim (L_\eta(p))^2 \cdot \frac{\pi_4(L_\varepsilon(p))}{\pi_4(L_\eta(p))} &\sim (L_\eta(p))^2 \cdot \mathbb{P}\left(\partial R_{L_\eta(p)} \rightsquigarrow_{4,\sigma_a} \partial R_{L_\varepsilon(p)}\right) \\ &\sim (L_\eta(p))^2 \cdot \left(\frac{L_\varepsilon(p)}{L_\eta(p)}\right)^{-\alpha_4}.\end{aligned}$$

□

The relation $\nu = 2 - \alpha_4$ now follows since from the proposition we have

$$1 \sim |p - p_c| \cdot (L(p))^2 \cdot (L(p))^{-\alpha_4}.$$

β**: Critical Exponent for** θ**.** From the result on L, we can deduce the *critical exponent* for θ from the following:

Proposition. Uniformly in p for $p > p_c$,

$$\theta(p) = \mathbb{P}(0 \rightsquigarrow \infty) \sim \mathbb{P}\left(0 \rightsquigarrow \partial R_{L(p)}\right).$$

Proof. It is clear that $\mathbb{P}(0 \rightsquigarrow \infty) \leq \mathbb{P}\left(0 \rightsquigarrow \partial R_{L(p)}\right)$. For the opposite inequality let us relate $L(p)$ to the *rescaling lemma*: We have by definition of $L(p)$ that

$$C(L(p), L(p)) \geq 1 - \varepsilon_0,$$

so by RSW

$$\begin{aligned} C(2L(p),L(p)) &\geq C(\frac{3}{2}\cdot L(p),L(p))^2\cdot C(L(p),L(p)) \\ &\geq (1-\sqrt{\varepsilon}_0)^{3/2}\cdot(1-\varepsilon_0) \\ &= 1-\frac{3}{2}\sqrt{\varepsilon_0}+O(\varepsilon_0) \\ &\geq 1-\frac{1}{16}\lambda, \end{aligned}$$

for ε_0 *sufficiently small* and $\lambda=\lambda(\varepsilon_0)$ *appropriately* chosen. The restriction on ε_0 is *inconsequential* since we already know that all L_{ε_0}'s are *comparable.*

The *rescaling lemma* therefore implies that *for all* $k\geq 1$,

$$C(2^{k+1}L(p),2^kL(p))\geq 1-c\lambda^{2^k},\quad c=1/16.$$

Next we have that

$$\{0\rightsquigarrow\infty\}\supseteq\{0\rightsquigarrow\partial R_{L(p)}(0)\}\cap\{\exists\text{ blue circuit in }A_{L(p)/3,L(p)}(0)\}\cap\{\text{“overlapping rectangles”}\}$$

(where the first *horizontal* rectangle is such that it *overlaps* $A_{L(p)/2,L(p)}(0)$).

[picture annulus construction near 0 together with overlapping rectangles construction to infinity leading to the event $\{0\rightsquigarrow\infty\}$...]

So we obtain the estimate

$$\mathbb{P}(0\rightsquigarrow\infty)\geq\mathbb{P}\left(0\rightsquigarrow\partial R_{L(p)}\right)\cdot\mathcal{F}(\lambda)\cdot\prod_{k\geq 0}(1-c\lambda^{2^k}):=C\mathbb{P}\left(0\rightsquigarrow\partial R_{L(p)}\right)>0,$$

uniformly in p for $p > p_c$. Here $\mathcal{F}(\varepsilon_0) \leq \mathbb{P}\left(\exists \text{ blue circuit in } A_{L(p),L(p)/3}\right)$ can be bounded in terms of $C_p(L(p), L(p))$ $(\geq 1 - \varepsilon_0)$: by diving R_L into 9 *smaller squares* of side length $L/3$, we have that $C(L, L) \leq 9 \cdot C(L/3, L/3)$.

[picture of square divided into 9 smaller squares: any crossing of big square implies crossing of *some* smaller square...] □

Remark. We will later have use for the relation between ε_0, the *crossing probability*, and λ (the *rescaling parameter*) to define a *renormalized lattice* on the *subcritical* lattice such that the new model is *even more subcritical* than the original.

The relation $\beta = \nu\alpha_1 \;\; (= \alpha_1 \cdot (2 - \alpha_4))$ now follows since from the proposition we have

$$\theta(p) \sim \mathbb{P}\left(0 \rightsquigarrow \partial R_{L(p)}\right) \sim \mathbb{P}_{p_c}\left(0 \rightsquigarrow \partial R_{L(p)}\right) = \pi_1(L(p)) \sim (|p - p_c|^{\nu})^{\alpha_1}.$$

The Characteristic Length $\xi(p)$. Recall that we already have that *if* $p < p_c$ (so that $p^* > p_c$ and the *characteristic length* $L(p^*)$ can be used as the *smallest scale* starting at which *rescaling* can be done, as in the proof of the previous proposition) then

$$\frac{1}{L(p^*)} \lesssim m(p) \lesssim \left(\frac{1}{L(p^*)} + \frac{\log L(p^*)}{L(p^*)}\right).$$

We will be able to complete the "picture" by showing that

$$L(p) \sim \xi(p) = \left(\frac{1}{\mathbb{E}_p(|\mathcal{C}(0)|; |\mathcal{C}(0)| < \infty)} \cdot \sum_x |x|^2 \cdot \mathbb{P}(T_{0x} \cap \mathscr{F}(\infty))\right)^{1/2}.$$

Remark. Note that since

$$\mathbb{E}(|\mathcal{C}(0)|; |\mathcal{C}(0)| < \infty) = \sum_x \mathbb{P}(T_{0x} \cap \mathscr{F}(\infty)),$$

$\xi(p)$ is nothing other than the (square root of the) *second moment* of the *probability measure* on $\mathbb{Z}^2$ *supported* on *lattice sites* given as

$$\mu(x) = \frac{\mathbb{P}(T_{0x} \cap \mathscr{F}(\infty))}{\sum_x \mathbb{P}(T_{0x} \cap \mathscr{F}(\infty))} \; .$$

First we will consider a *finitary moment* (i.e., summed over a *finite* box of scale *less* than the *characteristic length*) and relate it to π_1:

Lemma. Uniformly in $\mathbb{P}$ between $\mathbb{P}$ and $\mathbb{P}_{p^*}$ and $n \le L(p)$, we have for any $t \ge 0$,

$$\sum_{x \in R_n} |x|^t \cdot \tau_{0x} \;\sim\; \sum_{x \in R_n} |x|^t \cdot \tau'_{0x} \;\sim\; n^{t+2} \cdot \pi_1^2(n).$$

Here recall that

$$\tau'_{0x} = \mathbb{P}(0 \rightsquigarrow x \;\text{ inside } R_n) := \mathbb{P}(T'_{0x}).$$

Proof. First it is clear that

$$\sum_{x \in R_n} |x|^t \cdot \tau_{0x} \ge \sum_{x \in R_n} |x|^t \cdot \tau'_{0x},$$

so it remains to establish *two* more inequalities.

- $\sum_{x \in R_n} |x|^t \cdot \tau'_{0x} \gtrsim n^{2+t} \cdot \pi_1^2(n)$: Here we note that if $x \in R(n/3, 2n/3)$, then

$$T'_{0x} \supseteq \{0 \rightsquigarrow \partial R_n\} \;\cap\; \{\exists \text{ blue circuit in } A(2n/3, n)\} \;\cap\; \{x \rightsquigarrow \partial R_n\}.$$

[picture of R_n divided into three even subscales... the relevant connections and circuits... connection between 0 and x inside R_n in bold...]

The existence of a *circuit* has *probability* bounded *below* by some $\delta > 0$ which can be chosen *uniformly* in for $\mathbb{P}$ between $\mathbb{P}$ and $\mathbb{P}_{p^*}$ since we are *below* the *characteristic length*. Also, by *extendability* and the previous *variation of parameters* result, we also have that *uniformly*,

$$\mathbb{P}(0 \rightsquigarrow \partial R_n),\ \mathbb{P}(x \rightsquigarrow \partial R_n) \sim \pi_1(n).$$

We conclude therefore that

$$\tau'_{0x} \geq \delta \cdot \mathbb{P}(0 \rightsquigarrow \partial R_n) \cdot \mathbb{P}(x \rightsquigarrow \partial R_n) \gtrsim \pi_1(n)^2.$$

Altogether then

$$\sum_{x \in R_n} |x|^t \cdot \tau'_{0x} \ \gtrsim \sum_{x \in R(n/3, 2n/3)} |x|^t \cdot \pi_1^2(n) \ \gtrsim \ \frac{n^2}{3} \cdot \left(\frac{n}{3}\right)^t \cdot \pi_1^2(n) \sim n^{2+t} \cdot \pi_1^2(n).$$

- $\sum_{x \in R_n} |x|^t \cdot \tau_{0x} \lesssim n^{2+t} \cdot \pi_1^2(n)$: Here we will take a *logarithmic division* of R_n, i.e., let k be such that $2^k < n \leq 2^{k+1}$. Then we have

$$\sum_{x \in R_n} |x|^t \cdot \tau_{0x} \leq c + \sum_{j=3}^{k+1} \sum_{x \in A(2^{j-1}, 2^j)} |x|^t \cdot \tau_{0x},$$

with some constant $c > 0$ accounting for the *inner boundary* (near the *center*) terms. Now we observe that

$$x \in A(2^{j-1}, 2^j) \quad \Longrightarrow \quad R_{2^{j-2}}(0) \cap R_{2^{j-2}}(x) = \emptyset.$$

[picture of $x \in A(2^\ell, 2^{\ell+1})$ together with $R_{2^{\ell-1}}(0), R_{2^{\ell-1}}(x)$...]

Also,

$$\{0 \rightsquigarrow x\} \subseteq \{0 \rightsquigarrow \partial R_{2^{j-2}}(0)\} \ \cap \ \{x \rightsquigarrow \partial R_{2^{j-2}}(x)\},$$

so since the two squares are *disjoint*,

$$\tau_{0x} \leq \mathbb{P}(0 \rightsquigarrow \partial R_{2^{j-2}}(0)) \cdot \mathbb{P}(x \rightsquigarrow \partial R_{2^{j-2}}(x)) \lesssim \pi_1^2(2^{j-2}).$$

Using also that

$$|A(2^{j-1}, 2^j)| \lesssim 2^{2j}, \quad |x| \leq 2^j, \ x \in A(2^{j-1}, 2^j),$$

we may *reassemble* the sum as

$$\begin{aligned}\sum_{x \in R_n} |x|^t \cdot \tau_{0x} &\lesssim c + \sum_{j=3}^{k+1} 2^{2j} \cdot 2^{jt} \cdot \pi_2^2(2^{j-2}) \\ &\lesssim c + \pi_1^2(2^k) \cdot \sum_{j+3}^{k+1} 2^{j(2+t)} \cdot \frac{\pi_1^2(2^{j-2})}{\pi_1^2(2^k)} \\ &\lesssim \pi_1^2(2^k) \cdot 2^{(k+1)(2+t)} \cdot \sum_{\ell=0}^{\infty} 2^{-\ell(2+t)} \cdot \frac{\pi_1^2(2^{j-2})}{\pi_1^2(2^k)}.\end{aligned}$$

Finally, using *quasi–multiplicativity*, we have (with $\ell = k - j$)

$$\frac{\pi_1^2(2^{j-2})}{\pi_1^2(2^k)} \lesssim \pi_1^2(2^{j-2}, 2^k) \sim (2^{-\ell})^{2\alpha_1}.$$

Therefore, the sum *absolutely converges* to a *universal* constant and the remaining term is *comparable to* $n^{2+t} \cdot \pi_1^2(n)$.

□

Remark. It is worth emphasizing that fundamentally the above result is true because of a pair of inequalities:

- following from *extendability and RSW* we have

$$\tau'_{0x} \gtrsim \pi_1(n)^2, \quad \text{for } x \in A(n/3, 2n/3);$$

- by *locally reducing* the scale we have

$$\tau_{0x} \lesssim \pi_1^2(2^{\ell-2}), \quad \text{for } x \in A(2^{\ell-1}, 2^\ell).$$

A *renormalized* version of the second item will be used again shortly.

The result that $\xi(p) \sim L(p)$ as well as *critical exponent* for χ will follow once we extend the previous lemma to be a genuine *moment* estimate (i.e., summed over *all* lattice sites in $\mathbb{Z}^2$). Here, of course, we will have to restrict to the event $\mathscr{F}(\infty)$, otherwise the moment clearly diverges if $p > p_c$.

Lemma. For any $t \geq 0$, we have *uniformly* in $\mathbb{P}$ between $\mathbb{P}_p$ and $\mathbb{P}_{p^*}$ that

$$\sum_x |x|^t \cdot \mathbb{P}(T_{0x} \cap \mathscr{F}(\infty)) \sim L(p)^{t+2} \cdot \pi_1(L(p))^2.$$

Proof. We already have the *lower bound* from the previous lemma: let $L \equiv L(p)$, then the event $\mathscr{F}(\infty)$ can be replaced (as a *lower bound*) by the existence of *some yellow* circuit. More precisely, let $\delta > 0$ be yielded via RSW estimates so that

$$\mathbb{P}(A^*(L/3, L)) \geq \delta.$$

Again, δ can be chosen *uniformly in* $\mathbb{P}$ in the stated range since we are *below* the *characteristic length*. Then since

$$\mathbb{P}(\mathscr{F}(\infty)) \geq \mathbb{P}(A^*(L/3, L)),$$

[picture of yellow circuit in $A(L/3, L)$ preventing infinite cluster and forcing $C(0)$ to lie inside R_L...]

we have

$$\begin{aligned}\sum_{x\in R_L(0)} |x|^t \cdot \mathbb{P}(T_{0x} \cap \mathscr{F}(\infty)) &\geq \sum_{x\in R_{L/3}(0)} |x|^t \cdot \mathbb{P}(T_{0x} \cap \mathscr{F}(\infty)) \\ &\geq \sum_{x\in R_{L/3}(0)} |x|^t \cdot \mathbb{P}(T_{0x} \cap A^*(L/3, L)) \\ &\geq \sum_{x\in R_{L/3}(0)} |x|^t \cdot \mathbb{P}(T''_{0x} \cap A^*(L/3, L)) \\ &= \sum_{x\in R_{L/3}(0)} |x|^t \cdot \mathbb{P}(T''_{0x}) \cdot \mathbb{P}(A^*(L/3, L)) \\ &\geq \delta \cdot \sum_{x\in R_{L/3}(0)} |x|^t \cdot \tau''_{0x} \\ &\sim L^{t+2} \cdot \pi_1(L)^2.\end{aligned}$$

Here T''_{0x} denotes the event of a *connection inside* $R_{L/3}$ and the second equality is because of *independence*: the event T''_{0x} only depend on the *state inside* $R_{L/3}(0)$ whereas $A^*(L/3, L)$ depends on the state *outside* $R_{L/3}(0)$.

For the *upper bound* we will *tile* the plane with *translates* of $R_L(0)$ and *decompose* the sum to be a sum over such translates. We have from the previous lemma (and *translation invariance*) that for any *fixed* box it is the case that

$$\sum_{x\in R_L(0)+(m,n)} |x|^t \cdot \mathbb{P}(T_{0x} \cap \mathscr{F}(\infty)) \leq \sum_{x\in R_L(0)+(m,n)} |x|^t \cdot \tau_{0x} \sim L^{2+t} \cdot \pi_1^2(L).$$

[picture of plane covered by translates of $R_L(0)$...]

The result can therefore be finished by showing that translates *far away* contribute very little, i.e., *some exponential decay*. Let us first note the following estimate for τ_{0x}:

Proposition (α)**.** Let $x \in R_L(0) + (m,n)$, with $|(m,n)| = k \geq 4$. Then

$$\tau_{0x} \lesssim \pi_1^2(L) \cdot \mathbb{P}(\partial R_{2L}(0) \rightsquigarrow \partial R_{(k-1)L}(0)).$$

Proof. We have

$$T_{0x} \subseteq \{0 \rightsquigarrow \partial R_L(0)\} \cap \{\partial R_{2L}(0) \rightsquigarrow \partial R_{(k-1)L}(0)\} \cap \{x \rightsquigarrow \partial R_L(x)\},$$

and since all relevant boxes are *disjoint*, the result follows by *independence* and bounding $\{0 \rightsquigarrow \partial R_L(0)\}$ and $\{x \rightsquigarrow \partial R_L(x)\}$ by $\pi_1(L)$.

[picture of decomposing T_{0x} into two local connections at 0 and x together with a long connection...]

□

Note from the proposition that a *crude* bound via *subadditivity* and bounds on the *correlation function* would yield an *L–dependent boundary contribution*, i.e.,

$$\mathbb{P}(\partial R_{2L}(0) \rightsquigarrow \partial R_{(k-1)L}(0)) \lesssim (kL)^2 \cdot \tau_{0k}.$$

We can get rid of this L dependence via *renormalization*:

Proposition (Renormalization, ω)**.** Suppose $p < p_c$. Then

- There exists an *independent renormalized* percolation model at *parameter* $\mu(\varepsilon_0)$ whose sites consist of *1/9* of translates of $R_L(0)$ (periodically symmetrically *choose* one square out of each 9×9 block) so that for ε_0 *sufficiently small*,

$$\tau_{\mathcal{R}_0 \mathcal{R}_k} \leq \tau_{0k} \leq e^{-m(p) \cdot k}.$$

  Here $\mathcal{R}_k$ denotes a *generic translate* of $R_L(0)$ a distance (in the $\|\cdot\|_\infty$ norm) k away.

- *Conversely*, given any $\delta > 0$ (*small*) there exists some *fixed* $\Lambda(\delta) \gg 1$ of *order unity*, such that if

$$N = \Lambda \cdot L$$

and we consider the *independent percolation model* where translates of $R_N(0)$ are *declared yellow* (i.e., *open*) with probability $e^{-\Lambda}$, then for any $k > 0$,

$$\mathbb{P}(\exists x \in R_L : x \rightsquigarrow y \in R_{kL}) \leq \tau_{\mathcal{R}_1 \mathcal{R}_{k/\Lambda}}(e^{-\Lambda/9}) \lesssim e^{-c(\Lambda)k} + \delta^k,$$

for some constant which only depends on Λ.

Remark. The first item above is a *warm–up*. It is the estimate in the second item that is of importance to us: we have *scaled* and *renormalized* the original L (and therefore m) out of the problem, i.e., we can really think of a square of side length L as a *single site*.

Proof. (Renormalization like this is done in e.g., Reference 1, proof of Theorem 3.)

[picture of renormalized blocks with centers (the chosen ones) shaded...]

- We define a renormalized site $\mathcal{R}_k$ to be *yellow* (i.e., *open*) if there is *no blue* circuit surrounding it, so that

$$\mathbb{P}(\mathcal{R}_k \text{ yellow}) \equiv \mathbb{P}(A^*(L, 3L)^c).$$

Thining of the lattice (only taking 1 *block* out of 9) then ensures the model is *independent*; here we can define *connectivity* by letting the *chosen* block determine everything: e.g., if two *neighboring large* blocks have *yellow* centers then we consider them to be *connected*.

[picture of two connected neighboring renormalized blocks... may not be considered connected in the original model...]

By RSW, we can *lower* bound the existence of a *circuit* via RSW in the dual (*supercritical*) model at parameter $p^* = 1 - p$ so

$$\mathbb{P}(A(L, 3L)^c) \leq: 1 - \mathcal{F}(\varepsilon_0) := \mu.$$

Recall that by the definition of the *characteristic length* we have

$$C_{p^*}(L, L) \geq 1 - \varepsilon_0,$$

so for ε_0 *sufficiently small* (and hence the corresponding L_{ε_0} *sufficiently large*) $\mathcal{F}(\varepsilon_0)$ can be made arbitrarily small and in particular, we can choose ε_0 so that

$$\mu \leq p.$$

Thus the *renormalized* model is *more subcritical* than the original and the first item readily follows.

- Let us write $N = \Lambda \cdot L \equiv 2^\ell \cdot L$ for some $\ell > 0$.
  - A consequence of RSW and the *rescaling lemma* (we have already observed that for ε_0 sufficiently small, the (dual) *supercritical* model satisfies the *rescaling hypothesis* at scale L) is that

    $$A(N, 3N) \gtrsim 1 - c\lambda^{2^\ell}.$$

    Therefore, defining a *renormalized site* to be *open* at parameter μ if it is *not* surrounded by an annulus, as in the first item, we have that

    $$\mu \leq e^{-\Lambda}.$$
  - Next let us first define *connectivity* in the *entire renormalized* lattice (i.e., without

taking the factor of 1/9) as being connected by a path in the *original lattice*: i.e.,

$$\mathcal{R}_k \rightsquigarrow_\diamond \mathcal{R}_\ell \quad \text{iff} \quad \exists x \in \mathcal{R}_k, y \in \mathcal{R}_\ell, \quad \{x \rightsquigarrow y\}.$$

Then it is certainly the case that

$$\{\exists x \in R_L : x \rightsquigarrow y \in R_{kL}\} \subseteq \{\mathcal{R}_1 \rightsquigarrow_\diamond \mathcal{R}_{k-1}\}.$$

There are *9 ways* to define a sublattice of the *renormalized lattice* so that the resulting lattice is *independent* (as in the first item) and *given* any path between x and y as in the above event, *there exists* a *sublattice* defining an *independent* model which contains *at least* 1/9 of the total number of *blocks visited* by the path. Therefore,

$$\mathbb{P}(\exists x \in R_L : x \rightsquigarrow y \in R_{kL}) \leq \sum_{\omega : \mathcal{R}_1 \rightsquigarrow_\diamond R_{k-1}} \mu^{|\omega|/9} = \sum_{\omega : \mathcal{R}_1 \rightsquigarrow_\diamond R_{k-1}} (\mu^{1/9})^{|\omega|},$$

where the sum is over all *renormalized* paths between $\mathcal{R}_1$ and $\mathcal{R}_{k-1}$ and $|\omega|$ is the number of *renormalized blocks* which are considered *connected and open.*

[picture renormalized lattice as being connected via connection in the original lattice... in dash indicating that if renormalized blocks are connected this way, then there *cannot* be a *dual* circuit surrounding any sites in the "chosen" sublattice...

- Given any $k > 0$, we can choose $\mu(\delta)$ *sufficiently small* and $M = M_k(\delta)$ so that

$$(\mu^{1/9})^{M+1} < \delta^k, \quad \text{and} \quad \frac{1}{(1-\mu^{1/9})^{M-k}} = 1 + (M-k)\cdot\mu^{1/9} + O(\mu^{2/9}) \leq 2.$$

Indeed, it is sufficient to take (assuming $\delta < 1$)

$$\mu(\delta)^{1/9} = \delta^A \quad \text{for some} \quad A > 1 \quad \text{and} \quad k < M_k(\delta) < \delta^{-A} + k.$$

Then noting that a connection between $\mathcal{R}_1$ and $\mathcal{R}_{k-1}$ requires *at least* k occupied *renormalized blocks* and *truncating* the sum at M, we have

$$\sum_{\omega:\mathcal{R}_1 \rightsquigarrow_\diamond R_{k-1}} (\mu^{1/9})^{|\omega|} \leq \frac{1}{(1-\mu^{1/9})^{M_k - k}} \cdot \sum_{k \leq \mathrm{rad}(\omega) < M_k} (\mu^{1/9})^{|\omega|} \cdot (1-\mu^{1/9})^{M_k - |\omega|} + \delta$$
$$\lesssim \tau'_{\mathcal{R}_1 \mathcal{R}_{k-1}}(\mu^{1/9}) + O(\delta).$$

Here to obtain the last $\lesssim$ we have *recognized* that the *summand* inside the penultimate display contains the *weights* for *independent percolation inside* R_M where we *declare* a *renormalized block* to be *occupied* (or *yellow*) with probability $\mu^{1/9}$ and this corresponds exactly to *connection inside* R_M.

- Now the *dual model* to the *renormalized model* at parameter $\mu^{1/9}$ is *supercritical*, so letting $\bar{L} = L((\mu^{1/9})^*)$ be its *characteristic length*, we have (with appropriate choice of the corresponding λ and ε) that

$$\tau_{\mathcal{R}_0 \mathcal{R}_k}(\mu^{1/9}) \leq e^{-k/\bar{L}}.$$

(Here we envision $k \gg \bar{L}$, and recall that this bound can be established by using the *rescaling lemma* together with e.g., the observation that T_{0k} can be *prevented* by *crossings* of *all four* L by $L/2$ rectangles surrounding the origin.)

[picture crossings of rectangles preventing T_{0k}...

- Finally, we have altogether that (we have $\mu = e^{-\Lambda}$ depending on δ, and $N = \Lambda \cdot L$

is the size of the rescaled block)

$$\mathbb{P}(\exists x \in R_L : x \rightsquigarrow y \in R_{kL}) \leq T_{\mathcal{R}_1 \mathcal{R}_{k/\Lambda}}(e^{-\Lambda/9}) + \delta^k$$
$$\lesssim e^{-k/(\Lambda \cdot \bar{L})} + \delta^k,$$

so $c(\Lambda) = 1/(\Lambda \cdot \bar{L})$.

□

We can now proceed with the proof. We will divide into two cases:

- $(p < p_c)$. Here we have (since $m(p) > 0$) exponential decay of *correlations*:

  $$\tau_{0x} \leq e^{-m(p)\cdot|x|}, \quad \textit{for all } x,$$

  so the *decay* is provided by the *correlation functions*.

  For $x \in R_L(0) + (m,n), \quad |(m,n)| = kL$, we can use Proposition (α) and the second item in Proposition (ω) to obtain that

  $$\tau_{0x} \lesssim \pi_1^2(L) \cdot k \cdot e^{-c(\Lambda)\cdot k} + \delta^k.$$

  Here the factor of k is the *length* of the *renormalized boundary*.

  Altogether summing over $k = |(m,n)|/L$, bounding $|x|$ by kL, and using the above estimate for all the (m,n)–translates (there are $O((kL)^2)$ of them) we have for the *subcritical* case:

  $$\sum_{k\geq 0} \sum_{\substack{|(m,n)|=k \\ x\in R_L(0)+(m,n)}} |x|^t \cdot \mathbb{P}(T_{0x} \cap \mathscr{F}(\infty)) \leq \sum_{k\geq 0} \sum_{\substack{|(m,n)|=k \\ x\in R_L(0)+(m,n)}} |x|^t \cdot \tau_{0x}$$
  $$\lesssim L^{2+t} \cdot \pi_1^2(L) \ \cdot \ \left(\sum_{k=0}^{\infty} k^{3+t} \cdot e^{-c(\Lambda)\cdot k} + \delta^k \right).$$

- $(p > p_c)$. In this case, we have instead that the *dual* model has *exponential decay*.

- We expect (at least *morally*) that the event $\mathscr{F}(\infty)$ should contribute the necessary decay. Let us consider

$$\begin{aligned}\mathbb{P}(\mathscr{F}'(kL)) &:= \mathbb{P}(\mathscr{F}(kL) \cap \{0 \rightsquigarrow \partial R_{kL}(0)\}) \\ &\leq \mathbb{P}(\exists \text{ a } \textit{yellow} \text{ circuit between 0 and some } y \in \partial R_{kL}(0)) \\ &:= A^*_{0k}.\end{aligned}$$

(Here the intersection with the event $\{0 \rightsquigarrow \partial R_{kL}(0)\}$ is to *force* the circuit to be *outide* R_L; i.e., the *radius* of $\mathcal{C}(0)$ is kL. Thanks to both Brian Simanek and Helge Krüger.)

- Now the probability of the event A^*_{0k} can be estimated as follows: let us first assume *without loss of generality* that the relevant y of interest lies in $R_L(0) + (m, n)$ with $m > n$. Any circuit containing the origin must cross the x*–axis* and also the y*–axis* so *given* any particular ω^* containing a *dual* circuit, let us define

$$\begin{aligned}\mathcal{X}(\omega^*) &= R_L(0) + (\mathfrak{m}(\omega^*), 0) \\ \mathcal{Y}(\omega^*) &= R_L(0) + (0, \mathfrak{n}(\omega^*)),\end{aligned}$$

so that both $\mathfrak{m}(\omega^*), \mathfrak{n}(\omega^*) \leq 0$ are *minimal* such that the *circuit* crosses the x and y–axis at some point inside $\mathcal{X}(\omega^*)$ and $\mathcal{Y}(\omega^*)$, respectively.

[picture of (generic, i.e., slightly complicated) circuit with $\mathfrak{m}(\omega^*)$ and $\mathfrak{n}(\omega^*)$ labeled...]

The event A^*_{0k} can therefore be *decomposed* as follows:

$$A^*_{0k} \subseteq \bigcup_{\mathcal{R}_k} \bigcup_{\mathcal{Y}\leq 1} \bigcup_{\mathcal{X}\leq 1} T_{\mathcal{Y}\mathcal{R}_k} \circ T_{\mathcal{R}_k \mathcal{X}} \circ T_{\mathcal{X}\mathcal{Y}}.$$

Here the first *union* over $\mathcal{R}_k$ is over *renormalized blocks* accounting for ∂R_{kL} in the original lattice and as before, $\circ$ denotes *disjoint* occurrence. Since $m > n$, we have e.g.,

$$|\mathcal{Y} - \mathcal{R}_k| \gtrsim k + |\mathcal{Y}|, \quad |\mathcal{R}_k - \mathcal{X}| \gtrsim k + |\mathcal{X}|, \quad |\mathcal{X} - \mathcal{Y}| \gtrsim \max\{|\mathcal{X}|, \mathcal{Y}|\},$$

where the distance is now with respect to the *renormalized* $\|\cdot\|_\infty$ norm. We thus obtain from the second item in Proposition (ω) that (again k is the *renormalized length* of $|\partial R_{kL}|$)

$$\begin{aligned} \mathbb{P}(A^*_{0k}) &\lesssim \sum_{\mathcal{Y}\leq 1} \sum_{\mathcal{X}\leq 1} k \cdot e^{-c(\Lambda)\cdot[\ (k+|\mathcal{Y}|)\ +\ (k+|\mathcal{X}|)\ +\ \max\{|\mathcal{X}|,|\mathcal{Y}|\ ]} + \delta^k \\ &\lesssim k \cdot e^{-c(\Lambda)\cdot k} + \delta^k. \end{aligned}$$

[picture of ∂R_{kL} "covered by" $O(k)$ squares of scale L...]

- Using the *FKG inequality* (applied to the intersection of the *increasing* event T_{0x} with the *decreasing* event A^*_{0k}) we have

$$\mathbb{P}(T_{0x} \cap \mathscr{F}'(kL)) \lesssim (k \cdot e^{-c(\Lambda)\cdot k} + \delta^k) \cdot \tau_{0x}.$$

- Finally we have from Proposition (α) that for $x \in R_L(0) + (m,n)$, $|(m,n)| = kL$,

$$\tau_{0x} \lesssim \pi_1^2(L) \cdot \mathbb{P}(\partial R_{2L}(0) \rightsquigarrow \partial R_{(k-1)L}(0)) \lesssim \pi_1^2(L),$$

that is, we have bounded the probability of the "long" connection by 1.

Now we can *reassemble* the sum over all *translates*: first note that

$$\bigcup_{k\geq 0} \mathscr{F}'(kL) = \mathscr{F}(\infty).$$

(Indeed, any *finite* $\mathcal{C}(0)$ must also have *finite* (integral, in the $\|\cdot\|_\infty$ norm) radius.) So noting that

$$T_{_{0x}} \cap \mathcal{F}'(kL) = \emptyset, \quad \text{for } |x| > kL,$$

we now have for the *supercritical* case:

$$\begin{aligned}\sum_x |x|^t \cdot \mathbb{P}(T_{_{0x}} \cap \mathscr{F}(\infty)) &= \sum_{k\geq 0} \sum_{\substack{x\in R_L(0)+(m,n)\\ |(m,n)|=k'L;\ k'\leq k}} |x|^t \cdot \mathbb{P}(T_{_{0x}} \cap \mathscr{F}'(kL)) \\ &\lesssim L^{2+t} \cdot \pi_1^2(L) \cdot \sum_{k\geq 0} (k \cdot e^{-c(\Lambda)\cdot k} + \delta^k) \cdot \left(\sum_{k'\leq k} (k')^{2+t}\right) \\ &\lesssim L^{2+t} \cdot \pi_1^2(L) \cdot \left(\sum_{k\geq 0} k^{3+t} \cdot e^{-c(\Lambda)\cdot k} + k^{2+t} \cdot \delta^k\right).\end{aligned}$$

Finally, we will justify (reiterate) why the estimate is uniform in the range stated: in both cases, the non–trivial part of the final constant takes the *form* of the *convergent* sum

$$\sum_{k\geq 0} k^{3+t} \cdot e^{-c(\Lambda)\cdot k},$$

with the constant $c(\Lambda)$ a *universal constant* which is chosen only dependent on δ (in particular it is *independent* of the *length scales* $L(p)$ and $m(p)$ and thus independent of p). We have uniformity *only in* $\mathbb{P}$ between $\mathbb{P}_p$ and $\mathbb{P}_{p^*}$ since the previous lemma, which provides the *lower bound*, only holds uniformly in that range as it uses *quasi–multiplicativity*, etc. □

γ: Critical Exponent for χ.

Corollary. $\chi(p) = \mathbb{E}_p(|\mathcal{C}(0)|; |\mathcal{C}(0)| < \infty) \sim L(p)^2 \cdot \pi_1^2(L(p))$.

Proof. This is the last lemma with $t = 0$. □

The relation $\gamma = 2\nu \cdot (1 - \alpha_1) = 2(1 - \alpha_1) \cdot (2 - \alpha_4)$ now follows since from the corollary

$$\chi(p) \sim (L(p))^2 \cdot (L(p))^{-2\alpha_1}.$$

Finally, we also now have the *equivalence of* $\xi(p)$ *and* $L(p)$:

Corollary. $\xi(p) \sim L(p)$.

Proof. The last lemma with $t = 2$ gives that

$$\sum_x |x|^2 \cdot \mathbb{P}(0 \rightsquigarrow x, |\mathcal{C}(0)| < \infty) \sim L(p)^4 \cdot \pi_1(L(p))^2.$$

Thus together with the conclusion at $t = 0$, we have

$$\xi(p) \sim \left(\frac{L(p)^4 \cdot \pi_1(L(p)^2}{\mathbb{E}_p(|\mathcal{C}(0)|; |\mathcal{C}(0)| < \infty)} \right)^{1/2} \sim \left(\frac{L(p)^4 \cdot \pi_1(L(p)^2}{L(p)^2 \cdot \pi_1^2(L(p)} \right)^{1/2} = L(p).$$

□

References.

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2. *Scaling Relations for 2D–Percolation* by Harry Kesten. Commun. Math. Phys. **109**, 109–156 (1987).

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