VII. Variation of Parameter

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To facilitate the description of behavior *near criticality* we now show by appropriate use of *Russo's Formula* that the *multi–arm* exponents remain of the same order as *at criticality*, provided that we do not exceed the *characteristic length*.

Theorem. *Consider a* self–dual *percolation model. Then for*

$$n < N < L(p), \quad (j,\sigma) = (1,B) \quad \textit{and} \quad (j,\sigma) = (1,BYBY),$$

we have

$$\mathbb{P}_0(A_{j,\sigma}(n,N)) \sim_{j,\sigma} \mathbb{P}_1(A_{j,\sigma}(n,N)),$$

uniformly *for* $\mathbb{P}_0, \mathbb{P}_1$ *between (in the sense of* stochastic domination*)* $\mathbb{P}_p$ *and* $\mathbb{P}_{p^*}$*.*

A few observations before we begin the proof:

- Taking $\mathbb{P}_0 = \mathbb{P}_{p_c}$ we see that indeed, $\mathbb{P}_p(A_{j,\sigma}(n,N))$ behaves *critically* provided the lengths scales are *below* $L(p)$.

- Let us recall the definition of the *characteristic length*: let $s > 0$ be such that at $p = p_c$, we have *for all* L

  $$1 - s' \leq C_{p_c}(L,L) \leq s;$$

  let

  $$\varepsilon_0 < \min\{1 - s', s\}.$$

  Then we define the *characteristic length* by

  $$L(p) = \inf_n \{C_p(n,n) \leq \varepsilon_0\}, \quad p < p_c;$$

  $$L(p) = \inf_n \{C^*_p(n,n) \leq \varepsilon_0\}, \quad p > p_c,$$

so that

$$L(p) \nearrow \infty \quad \text{as} \quad p \to p_c.$$

- Note then that if a model is *self–dual*, i.e., $p_c = p_c^*$, then

$$L(p) = L(p^*).$$

  Indeed, if e.g., $p < p_c$, then $p^* > p_c^* = p_c$ and therefore the definitions of characteristic lengths *directly coincide.*

- It also follows that if p_t is *between* p *and* p^*, then

$$L(p_t) \geq L(p) = L(p^*).$$

  Indeed, if e.g., $p < p_c$, then $p^* > p_c$ so that $p_t \in (p, p^*)$ and so by *(stochastic) monotonicity* $L(p_t) \geq L(p)$. It follows that in the context of the theorem, a *single characteristic length* $L(p)$ *governs all* $\mathbb{P}_0, \mathbb{P}_1$ between $\mathbb{P}_p$ and $\mathbb{P}_{p^*}$, in the sense that provided we stay *below this length scale*, we may perform RSW constructions, etc., *uniformly in* p for all $p \in (p, p^*)$.

Remark. From the last item we see that if the model were *not self–dual*, then we ought to consider a characteristic length

$$L' = \min\{L(p), L(p^*)\}.$$

One–Arm. For simplicity let us first describe how one accomplishes this for $A_{1,B}$ the *long way* via *estimation* of *four–arm* events (of course, here the result can easily follow by *monotonicity*):

- Recall that *Russo's Formula* for *increasing* functions says that

$$\frac{d}{dp}\mathbb{P}_p(A) = \mathbb{E}_p(|\delta A|),$$

  where δA is the set of *pivotal edges* of the event A.

- The key observation is then that *locally* near any *pivotal edge* for $A_{1,B}$ there *emanates four alternating arms* (just as in the case of the event of a *left–right crossing*).

- Consequently, roughly speaking the *event* of v being *pivotal* for $A_{1,B}(n,N)$ can be decomposed into 3 pieces:

  i) a connection from near the *origin* to v;

  ii) an *alternating four arm event* near v;

  iii) a connection between the *vicinity of* v and *boundary of the original* box.

[picture of point v on one long arm from 0 to right boundary together with yellow arms to top and bottom indicating v is pivotal; small square around v: locally four arms...]

- By *quasi–multiplicativity* and *extendability* items i) and iii) can be combined and *bounded by the original event* $A(n,N)$, so that we are left with an *upper bound* of the form

$$\begin{aligned}\frac{d}{dp}\mathbb{P}_p(A_{1,B}(n,N)) &= \sum_{v\in R_N}\mathbb{P}_p(v \text{ pivotal for } A_{1,B}(n,N))\\ &\lesssim \sum_{v\in R_N}\mathbb{P}_p(A_{1,B}(n,N))\cdot\mathbb{P}_p(v\rightsquigarrow_{4,\sigma_a}\partial R_{m_v}(v)),\end{aligned}$$

for some $m_v \sim d(0,v)$ which represents the *vicinity of* v. Here σ_a denotes the *alternating* 4–arm configuration. Let us dispense with some *technical considerations:*

  - We will parameterize the change in p via t:

$$p(t) = tp_1 + (1-t)p_0,$$

so that

$$p'(t) = p_1 - p_0.$$

By the *chain rule* (if k indexes the relevant edges/sites, then $\frac{d}{dt} = \sum_k \frac{dp}{dt} \cdot \frac{\partial}{\partial p_k}$) we can recast the above estimate as

$$\begin{aligned}\frac{d}{dt}\mathbb{P}_{p(t)}(A_{1,B}(n,N)) &\lesssim \sum_{v \in R_N} \frac{dp}{dt} \cdot \mathbb{P}_p(A_{1,B}(n,N)) \cdot \mathbb{P}_p(v \rightsquigarrow_{4,\sigma_a} \partial R_{m_v}(v)) \\ &= \sum_{v \in R_N} (p_1 - p_0) \cdot \mathbb{P}_p(A_{1,B}(n,N)) \cdot \mathbb{P}_p(v \rightsquigarrow_{4,\sigma_a} \partial R_{m_v}(v)).\end{aligned}$$

- For simplicity *overall* we are considering *homogeneous* models, i.e., the parameter p is *spatially homogeneous*, but the arguments here can be adapted to tolerate some *inhomogeneity*, *provided* the resulting measure remains between $\mathbb{P}_p$ and $\mathbb{P}_{p^*}$.
- The previous observation is useful for us in the following way: we would like to ensure that m_v (the vicinity of a *pivotal site*) is sufficiently "large" so that we can reasonably estimate the *4–arm* event.

  To this end let us set

$$N = 2^K, n = 2^k, \quad N' = 2^{K-4}, n' = 2^{k+3}$$

  and define

$$\tilde{p}_v(t) = \begin{cases} p(t) & \text{if } v \in R_{N'} \setminus R_{n'} \\ p(0) & \text{if } v \in \text{“boundary layers”} \quad (R_N \setminus R_{N'} \cup R_n \setminus R_{n'}), \end{cases}$$

  that is, we permit the parameter p to be *constant* and equal to p_0 in a boundary layer around the *annulus* $A(n,N)$. So the corresponding measures $P_{\tilde{p}(t)}$ *interpolate* between $\mathbb{P}_0$ and $\mathbb{P}_{\tilde{p}(1)}$ which *coincides with* $\mathbb{P}_1$ inside $R'_N \setminus R'_n$.

[picture of annulus with "boundary layer" and labels...]

But *quasi–multiplicativity* can still be obtained for $\mathbb{P}_{\tilde{p}(1)}$ and so *if* we show $\mathbb{P}_{\tilde{p}(1)} \sim \mathbb{P}_0$, then we would have, together *quasi–multiplicativity* for $\mathbb{P}_0$ that

$$\mathbb{P}_1(n', N') \sim \mathbb{P}_{\tilde{p}(1)}(n', N') \sim \mathbb{P}_{\tilde{p}(1)}(n, N) \sim \mathbb{P}_0(n, N) \sim \mathbb{P}_0(n', N').$$

We may then return to (n, N) with one more application of *quasi–multiplicativity* for $\mathbb{P}_0$ and $\mathbb{P}_1$. Thus it is sufficient to work with the $\mathbb{P}_{\tilde{p}(t)}$'s.

- Since $\tilde{p}(t)$ is constant in the "boundary layer", the corresponding *Russo's Formula* expression *does not contain* terms involving v's too close to the boundary. From now one we will suppress $\tilde{}$ and consider the estimate

$$\frac{d}{dt}\mathbb{P}_{p(t)}(A_{1,B}(n, N)) \lesssim \sum_{v \in A(2^{k+3}, 2^{K-4})} (p(1) - p_0) \cdot \mathbb{P}_p(A_{1,B}(n, N)) \cdot \mathbb{P}_p(v \rightsquigarrow_{4,\sigma_a} \partial R_{m_v}(v)).$$

- Dividing by $\mathbb{P}_t(A_{1,B}(n, N))$ we obtain an estimate on the *logarithmic derivative*:

$$\frac{d}{dt}\log[\mathbb{P}_t(A_{1,B}(n, N))] \lesssim \sum_{v \in A(2^{k+3}, 2^{K-4})} (p(1) - p_0) \cdot \mathbb{P}_t(A_{4,\sigma_a}(0, m_v)).$$

We therefore must show that

$$\sum_{v \in A(2^{k+3}, 2^{K-4})} (p(1) - p_0) \cdot \mathbb{P}_t(A_{4,\sigma_a}(0, m_v)) < \infty.$$

- Next note that *Russo's Formula* gives (recall that A_{4,σ_a} at v corresponds to v being *pivotal* for the event of a *left right crossing*) for any n the bound:

$$\int_0^1 \sum_{v \in R_n} (p_1 - p_0) \cdot \mathbb{P}_t(A_{4,\sigma_a}(0, n))\ dt = \mathbb{P}_1(\mathcal{C}(S_n)) - \mathbb{P}_0(\mathcal{C}(S_n)) \leq 1.$$

We can use this estimate the following way:

- *Assuming* that

$$\mathbb{P}_t(A_{4,\sigma_a}(0, n)) \sim \pi_4(n) \sim n^{-\alpha_4},$$

(this is in essence what we are trying to prove) where $\alpha_4 > 0$ denotes the *critical four–arm exponent*, we obtain that

$$(p - p_c) \cdot n^2 \cdot \pi_4(n) := C_0 \leq 1.$$

- Then if we consider $2^K \sim N$ and a *logarithmic annular estimate* (going inwards, applying ignoring *boundary effects* for now) using the last estimate to bound the *outermost term* and assuming all 4–arm events can be described by α_4, then we may recast the estimate as

$$\begin{aligned}
&\int_0^1 \sum_{v \in R_N} (p_1 - p_0) \cdot \mathbb{P}_t(A_{4,\sigma_a}(0, m_v)) \, dt \\
&\lesssim_p C_0 \cdot \sum_{2^{\ell-1} \leq d(0,v) \leq 2^\ell} 2^{\alpha_4} \cdot \frac{\#\{\text{vertices in } A(2^{\ell-1}, 2^\ell)\}}{\#\{\text{vertices in } A(2^{N-1}, 2^N)\}} \\
&\lesssim \sum_\ell (2^{\alpha_4 - 2})^\ell
\end{aligned}$$

which converges *provided that* $2^{\alpha_4 - 2} < 1 \iff \alpha_4 < 2$.

- What we will actually do is estimate the four arm event in terms of the *five–arm* event by using *Reimer's inequality* in reverse and the *existential* exponent for *one–arm*. The *weaker* statement

$$\sum_{v \in R_{n/2}} \mathbb{P}_p(v \rightsquigarrow_{5,\sigma} \partial R_n) \sim 1, \quad \text{uniformly in } p\text{, provided } n < L(p)$$

will suffice for us:

  - on the one hand it is a *combinatorial fact* that (with $\sigma = BYBBY$) for *any* measure $\mathbb{P}$,

$$\sum_{v \in R_{n/2}} \mathbb{P}(v \rightsquigarrow_{5,\sigma} \partial R_n) = \mathbb{P}\left(\cup_{v \in R_{n/2}} \{v \rightsquigarrow_{5,\sigma} \partial R_n\}\right) \leq 1;$$

- on the other hand, by RSW we have that there exists $C > 0$ such that

$$\mathbb{P}_{p'}\left(\cup_{v \in R_{n/2}}\{v \rightsquigarrow_{5,\sigma} \partial R_n\}\right) \geq C,$$

  where, provided that $n < L(p)$, the constant C can be made uniform in p' between p and p^* by careful choice of the parameter governing bounds on the *crossing probabilities* in the definition of the *characteristic length*.

- Suppose then that we have a *pivotal site*

$$v \in A(2^\ell, 2^{\ell+1}), \quad k+3 \leq \ell \leq K-4.$$

For such a site v, we may take

$$m_v = 2^\ell.$$

The *annulus* $A(2^\ell, 2^{\ell+1})$ can be decomposed in *12 smaller squares*:

$$A(2^\ell, 2^{\ell+1}) = R_1^{(\ell)} \cup \cdots \cup R_{12}^{(\ell)},$$

where $R_j^{(\ell)}$'s are squares of side–length 2^ℓ. Let us denote by $R(v)$ the square which *contains* v.

[picture of annulus decomposed into 12 smaller squares and v lying in one of them...]

Direct *inspection* then shows that it is the case that

$$\{v \rightsquigarrow_{4,\sigma_a} \partial R_{2^{\ell+1}}(v)\} \subseteq \{v \rightsquigarrow_{4,\sigma_a} \partial R'(v)\} \subseteq \{v \rightsquigarrow_{4,\sigma_a} \partial R_{2^{\ell-1}}(v)\},$$

where $R'(v) = \frac{3}{2} \cdot R(v)$.

[picture of $R(v)$ with v close to e.g., the right boundary ("extremal case") together with $R'(v), R_{2^{\ell-1}}(v)$ and $R_{2^{\ell+1}}(v)$...]

Since for each t, we have

$$\mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R_{2^{\ell+1}}(v)) \sim \mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R_{2^{\ell-1}}(v)),$$
$$\mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R(v)) \sim \mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R'(v)),$$

with a constant which can be made *uniform in* t, we have that

$$\mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R_{2^\ell})) \sim \mathbb{P}_t(v \rightsquigarrow \partial R(v)),$$

so it is sufficient to establish the estimate

$$\begin{aligned} &\sum_{\ell=k+3}^{K-4} \sum_{k=1}^{12} \sum_{v \in S_j^{(\ell)}} (p(1)-p_0) \cdot \mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R_j^{(\ell)}) \\ :=& \sum_{\ell=k+3}^{K-4} \sum_{j=1}^{12} (p(1)-p_0) \cdot S_j^{(\ell)}(t) \\ <& \infty. \end{aligned}$$

[picture logarithmic annuli divided into smaller squares of *various* scales, with "small boundary layer" shaded (off scale)...]

- Next we show that $S_j^{(\ell)}(t)$ decays with ℓ^{-1}:
  - let us first recall that from *Russo's Formula* we have

$$|\mathcal{C}_{p(1)}(2^K,2^K) - \mathcal{C}_{p_0}(2^K,2^K)| = |\sum_{v \in R_{2K-3}} \int_0^1 (p(1)-p_0) \cdot \mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R_{2^K})\, dt \,| \leq 1.$$

    (This is the expression we have *before* the *logarithmic* derivative.) Since we have by *quasi–multiplicativity* that

$$\mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R_{2^K}) \sim \mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R(v)) \cdot \mathbb{P}_t(\partial R(v) \rightsquigarrow_{4,\sigma_a} \partial R_{2^K}),$$

together with the previous display we have

$$\begin{aligned} 1 \geq (p(1) - p_0) \cdot \sum_{v \in R_{2K-3}} \mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R_{2^K}) \\ \sim \sum_{v \in R_{2K-3}} \mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R(v)) \cdot \mathbb{P}_t(\partial R(v) \rightsquigarrow_{4,\sigma_a} \partial R_{2^K}). \end{aligned}$$

Next we have that again by *extendability* and *translation invariance*,

$$\mathbb{P}_t(\partial R(v) \rightsquigarrow_{4,\sigma_a} \partial R_{2^K}) \sim \mathbb{P}_t(\partial R(w) \rightsquigarrow_{4,\sigma_a} \partial R_{2^K})$$

for any v, w in the *same* annulus $A(2^\ell, 2^{\ell+1})$. So letting $R^{(\ell)}$ denote *some* (any) square comprising $A(2^\ell, 2^{\ell+1})$ and now writing

$$S^{(\ell)}(t) = \sum_{j,\ v \in R_j^{(\ell)}} \mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R(v)),$$

we have

$$1 \geq \sum_{v \in R_{2K-3}} \mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R_{2^K}) \sim \sum_{\ell=k+3}^{K-4} S^{(\ell)}(t) \cdot \mathbb{P}(\partial R^{(\ell)} \rightsquigarrow R_{2^K}).$$

- Now from *Reimer's inequality* we have (with $\sigma = BBYBY$)

$$\mathbb{P}_t(\partial R^{(\ell)} \rightsquigarrow_{5,\sigma} \partial R_{2^K}) \cdot \mathbb{P}_t(\partial R^{(\ell)} \rightsquigarrow_{1,B} \partial R_{2^K})^{-1} \leq \mathbb{P}_t(\partial R^{(\ell)} \rightsquigarrow_{4,\sigma} \partial R_{2^K}),$$

so using the *existential* bound for one–arm

$$\mathbb{P}_t(A_1(n, N)) \lesssim \left(\frac{n}{N}\right)^{\alpha'}, \quad \textit{some } \alpha' > 0,$$

and summing over ℓ, we obtain the expression

$$
\begin{aligned}
1 &\geq \sum_{v \in R_{2^{K-3}}} \mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R_{2^K}) \\
&\sim \sum_{\ell=k+3}^{K-4} S^{(\ell)}(t) \cdot \mathbb{P}_t(\partial R^{(\ell)} \rightsquigarrow_{4,\sigma_a} \partial R_{2^K}) \\
&\geq \sum_{\ell=k+3}^{K-4} S^{(\ell)}(t) \cdot [\mathbb{P}_t(\partial R^{(\ell)} \rightsquigarrow_{5,\sigma} \partial R_{2^K}) \cdot \mathbb{P}_t(\partial R^{(\ell)} \rightsquigarrow_{1,B} \partial R_{2^K})^{-1}] \\
&\gtrsim \sum_{\ell=k+3}^{K-4} S^{(\ell)}(t) \cdot 2^{\alpha'(K-\ell)} \cdot \mathbb{P}_t(\partial R^{(\ell)} \rightsquigarrow_{5,\sigma} \partial R_{2^K}).
\end{aligned}
$$

- In order to estimate the size of an *individual*

$$S^{(\ell_*)}(t) \sim 12 \cdot S_j^{(\ell_*)}, \quad \text{any } j = 1, 2, \ldots, 12,$$

let us fix some $\ell = \ell_*$ and redo the estimate with

$$m_v = \ell_*, \quad \textit{for all } v \in R_{2^{K-3}}.$$

By *translation invariance* the term $S_j^{(\ell_*)}$ can be *pulled out* of the sum over all boxes, so the resulting estimate becomes

$$1 \gtrsim S^{(\ell_*)}(t) \cdot 2^{\alpha'(K-\ell_*)} \cdot \sum_{R^{(\ell_*)} \subseteq A(2^{k+3}, 2^{K-4})} \mathbb{P}_t(\partial R^{(\ell_*)} \rightsquigarrow_{5,\sigma} \partial R_{2^K}).$$

[picture R_{2^K} divided into squares of the same size with original logarithmic division lightly drawn...]

- Next we note that defining

$$\tilde{S}^{(\ell_*)}(t) = \sum_{v \in R^{(\ell_*)}} \mathbb{P}_t(v \rightsquigarrow_{5,\sigma_a} \partial R^{(\ell_*)}(v))]$$

to be the *corresponding sum* for *five arm events*, then by the *same reasoning* as for the $S_j^{(\ell)}$'s, we obtain that

$$\sum_{v \in R_{2^{K-3}}} \mathbb{P}_t(v \rightsquigarrow_{5,\sigma} \partial R_{2^K}) \sim \sum_{R^{(\ell_*)} \subseteq A(2^{k+3}, 2^{K-4})} \tilde{S}^{(\ell_*)}(t) \cdot \mathbb{P}_t(\partial R^{(\ell_*)} \rightsquigarrow_{5,\sigma} \partial R_{2^K}),$$

but by the *universal result* on five arm events, we have that *both* the *left hand side and* $\tilde{S}^{(\ell_*)}$ $\left(= \sum_{v \in R^{(\ell_*)}} \mathbb{P}_t(v \rightsquigarrow_{5,\sigma_a} \partial R^{(\ell_*)}), \quad \textit{some (any) } R_j^{(\ell)}\right)$ are of *order unity* and therefore

$$\sum_{R^{(\ell_*)} \subseteq A(2^{k+3}, 2^{K-4})} \mathbb{P}_t(\partial R^{(\ell)} \rightsquigarrow_{5,\sigma_a} \partial R_{2^K}) \sim 1.$$

(Indeed, this is just a *renormalized* version of the *five–arm* result, where we consider a *coarsened lattice* with blocks of scale ℓ_*.)

[picture of renormalized lattice of scale 2^{K-4} inside R_{2^K}... ∂R_{2^K} pretty far away...]

- *Combining* the last two items we arrive at the *estimate* that *uniformly in* t,

$$S^{(\ell_*)}(t) \lesssim 2^{-\alpha' \cdot (K - \ell_*)},$$

so finally

$$\frac{d}{dt} \log[\mathbb{P}_t(A_{1,B}(n,N)] \lesssim \sum_{v \in A(2^{k+3}, 2^{K-4})} (p_1 - p_0) \cdot \mathbb{P}_t(A_{4,\sigma_a}(0, m_v))$$

$$\sim \sum_{\ell = k+3}^{K-4} S^{(\ell)}(t) \lesssim \sum_{\ell = k+3}^{K-4} 2^{-\alpha' \cdot (K - \ell)} < \infty.$$

Generalized Russo's Formula. For multi–arm events, we will need to consider intersections of an *increasing* and *decreasing* event.

Lemma. Let A^+, A^- be *monotonone* increasing and decreasing events (respectively) which depend on *finitely many* sites (or edges) R. Let $p : t \in [0,1] \to [0,1]$ by differentiable and let $\mathbb{P}_t = \mathbb{P}_{p(t)}$. Then indexing the sites (or edges) by e_k and writing (in *binary*) e.g.,

$$D_k^{10} = \{e_k \in \delta A^+\} \cap \{e_k \notin \delta A^-\}, \quad D_k^{01} = \{e_k \notin \delta A^+\} \cap \{e_k \in \delta A^-\},$$

we have

$$\frac{d}{dt}\mathbb{P}_t(A^+ \cap A^-) = \sum_{e_k \in R} \frac{dp}{dt}(t) \cdot \left[\mathbb{P}_t(D_k^{10} \cap A^-) - \mathbb{P}_t(D_k^{01} \cap A^+)\right].$$

Proof. This follows as in the proof of the usual *Russo's formula*. Indeed, we have the following *Bayesian decomposition*:

$$\begin{aligned}\mathbb{P}_t(A^+ \cap A^-) &= \mathbb{P}_t(D_k^{00}) \cdot \mathbb{P}_t(A^+ \cap A^- \mid D_k^{00}) + \mathbb{P}_t(D_k^{10}) \cdot \mathbb{P}_t(A^+ \cap A^- \mid D_k^{10}) \\ &\quad + \mathbb{P}_t(D_k^{01}) \cdot \mathbb{P}_t(A^+ \cap A^- \mid D_k^{01}) + \mathbb{P}_t(D_k^{11}) \cdot \mathbb{P}_t(A^+ \cap A^- \mid D_k^{11}).\end{aligned}$$

Now it is immediately clear that the *first* term does *not change* with $p_k(t)$ (the *parameter* at e_k at time t) whereas the *last* term is *identically zero* since e_k cannot be *blue and yellow simultaneously*. On the other hand, we have

$$\begin{aligned}\mathbb{P}_t(A^+ \cap A^- \mid D_k^{10}) &= \mathbb{P}_t(A^+ \mid A^- \cap D_k^{10}) \cdot \mathbb{P}_t(A^- \mid D_k^{10}) \\ &= p_k \cdot \mathbb{P}_t(A^- \mid D_k^{10}).\end{aligned}$$

Similarly,

$$\mathbb{P}_t(A^+ \cap A^- \mid D_k^{01}) = (1 - p_k) \cdot \mathbb{P}_t(A^+ \mid D_k^{01}).$$

Altogether we now have

$$\begin{aligned}\mathbb{P}_t(A^+ \cap A^-) &= p_k \cdot \mathbb{P}_t(D_k^{10}) \cdot \mathbb{P}_t(A^- \mid D_k^{10}) + (1 - p_k) \cdot \mathbb{P}_t(D_k^{01}) \cdot \mathbb{P}_t(A^+ \mid D_k^{01}) \\ &= p_k \cdot \mathbb{P}_t(D_k^{10} \cap A^-) + (1 - p_k) \cdot \mathbb{P}_t(D_k^{01} \cap A^+).\end{aligned}$$

Finally, noting that e.g., $D_k^{10} \cap A^-$ *does not* depend on p_k, differentiating yields the the result:

$$\frac{\partial}{\partial p_k}\mathbb{P}_t(A^+ \cap A^-) = \mathbb{P}_t(D_k^{10} \cap A^-) - \mathbb{P}_t(D_k^{01} \cap A^+).$$

□

Even Alternating–Arms. For j even and $\sigma = BYBYBY...$ alternating, we can estimate the *two terms* from the *generalized* Russo's formula *separately*: E.g., the event

$$\{v \in \delta A^+\} \cap \{v \notin \delta A^-\} \cap A^-$$

still *locally* leads to *four–arms* around v because of the event $\{v \in \delta A^+\}$. The only difference here is the *yellow* "pinning" arms to enforce the *pivotal nature* of v may "run into" *yellow* arms which *accomplish the event* A^- (the fact that σ is *alternating* means that the pinning arms always run into *yellow* arms before *blue*). Regardless, with e.g., $v \in A(2^\ell, 2^{\ell+1})$ (with $2^\ell \ll N$) there still does exist four *disjoint* arms to $\partial R_{2^\ell}(v)$.

[picture $A_4(n, N)$ with v pivotal with yellow arms locally at v "joining" the longer yellow arms...]

Therefore, it is still sufficient to estimate $\sum_{v \in A(2^{k+3}, 2^{K-4})} \mathbb{P}_t(v \rightsquigarrow_{4,\sigma_a} \partial R_{m_v}(v))$ as before.

References.

1. *Near Critical Percolation in Two Dimensions* by Pierre Nolin. Electronic Journal of Probability, Vol. 13, no. **55**, 1562–1623 (2008).

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