VI. Universal Arm Exponents

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From the *"fences and corridors"* constructions we already have that the (probabilities of) multiple crossings of an annulus admit description via certain exponents:

Lemma. Fix some $0 < p < 1$. For fixed $j \geq 1$ and color sequence σ, there exists exponents

$$0 < \alpha_j, \alpha < \infty$$

such that uniformly for all $\eta \geq \eta_0 > 0$

$$\left(\frac{n}{N}\right)^{\alpha_j} \lesssim_j \mathbb{P}_p(A_{j,\sigma}(n, N) \lesssim \left(\frac{n}{N}\right)^{\alpha}.$$

Here we will focus on the derivation of the *precise exponents* (i.e., the exponents appears in both the upper and lower bounds) in a few cases (in particular 5–arm crossings in a full annulus and 3–arm crossings in a half annulus). These cases can be deduced from "elementary" percolation estimates as we have been doing and can also be used to establish certain properties of relevant curves in the *scaling limit*. We state the results at *criticality* although as before they also hold *near criticality* provided the relevant length scale is *below* the *characteristic length*.

5–Arm Exponent in Full Space. Here we will establish that

Lemma. Suppose $p = p_c$. Let $\sigma = BYBBY$, then as $N \to \infty$,

$$A_{5,\sigma}(0, N) \sim N^{-2}.$$

Proof. The starting point is to note that we already have

$$\mathbb{P}_p(v \rightsquigarrow_{5,\sigma} \partial R_N) \sim \mathbb{P}_p(0 \rightsquigarrow_{5,\sigma} \partial R_N),$$

uniformly in N for $v \in R_{N/2}$. As there are $O(N^2)$ such points, it is sufficient to show that the number of *5–arm sites* is $O(1)$.

- (Upper bound) Here by a *relocation of arms argument*, it is also the case that up to yet another loss of constant, we may assume the *landings* of the arms on ∂R_N are such that v is a *pivotal site for a left right blue crossing* of R_N. Further, we may replace the entire event of interest by the event $A^I_{5,\sigma}$ by *prescribing landing intervals.*

  [picture of "rearranged" 5–arms with correct landing locations; extra black arm to top boundary...]

  Let

  $$A_v = \{v \rightsquigarrow^I_{5,\sigma} \partial R_N\} \cap \{v \text{ is blue}\},$$

  then

  $$\mathbb{P}_p(A_v) = p \cdot \mathbb{P}(v \rightsquigarrow_{5,\sigma} \partial R_N),$$

  so up to the constant p we may consider instead the event A_v where an actual *left right blue crossing* has occurred and the relevant arms land in *prescribed intervals.*

  We claim that (given a *configuration* ω) A_v can occur for *only one site* v in $R_{N/2}$: Suppose there is a site $w \neq v$ such that A_w also occurs (A_w must obey the same *landing interval prescriptions*) then:

  - w cannot be in any of the non–trivial regions determined by the crossings since in any such region w cannot even be a pivotal site for a left right crossing: one of the four required arms cannot occur;

- ◦ it remains the possibility that w lies on one of the blue arms; w cannot be on the *blue arm to the top* which is *not* part of the left right crossing since then it has no possibility of having a *yellow arm to the bottom*;

- ◦ finally, if w lies on the blue crossing itself, we may divide into two cases:

[picture of w to left of v and w to right of v with "impossible" crossings in dash...]

  - ∗ if w lies to the left of v, then the *yellow* arm to the *top* (in the correct *landing interval*) cannot occur without intersection the *blue* arm to the *top*;
  - ∗ whereas if w lies to the right of v, then the *blue* arm to the *top* cannot be accomplished without intersecting the *yellow* arm to the *top* already present.

  Since both possibilities are forbidden, $w = v$.

We now have the required upper bound:

$$\begin{aligned} 1 &\geq \mathbb{P}_p(\cup_{v \in R_{N/2}} A_v) \\ &= \sum_{v \in R_{N/2}} \mathbb{P}_p(A_v) \\ &\sim N^2 \cdot \mathbb{P}_p(0 \rightsquigarrow_{5,\sigma} \partial R_N). \end{aligned}$$

- (Lower bound) Here we show that the 5–arm event occurs with positive probability via RSW estimates.

  - ◦ First with probability $\delta^2 > 0$ there is a *blue left right crossing* in a *strip* immediately above the midline of R_N and a *yellow left right crossing* in the strip *below* the midline of R_N.

- Consider the blue crossing to be the lowest and denote it c so that all sites on it have a *yellow* connection to the *bottom*; note that the existence of the two crossing in the previous item (especially the existence of the *yellow crossing*) means that the lowest blue crossing is "approximately" near the center.

[picture of blue crossing strip above x–axis and yellow crossing in strip below...]

- Next construct two crossings from c to the *top* in two more "corridors", one *blue* (to the *left* of center) and one *yellow* (to the *right* of center) with landing points v_b and v_y.
- Following c from left to right, if v is the *last* vertex before v_y connected to the top by a *blue* connection, then the 5–arm event occurs at v: at v there must be another *yellow* connection to the top since otherwise there would be a *blue arc* connecting the crossing ending at v_1 to some point on the *right* of v, contradicting the definition of v.

[picture of 5–arm event occurring at v with blue and yellow crossings in relevant corridors and v_b, v, v_y labeled and possible "small" blue connection to some point to the right of v in the absence of a yellow connection to the top...]

Careful choice of scales ensures that such a v at which a 5–arm event occurs lies in $R_{N/2}$ so we have that for some constant C,

$$\mathbb{P}_p(\cup_{v\in R_{N/2}}\{v \rightsquigarrow_{5,\sigma} \partial R_N\}) \geq C > 0.$$

Altogether we have the lower bound:

$$\begin{aligned} C &\leq \sum_{v\in R_{N/2}} \mathbb{P}_p(v \rightsquigarrow_{5,\sigma} \partial R_N\}) \\ &\lesssim N^2 \cdot \mathbb{P}_p(0 \rightsquigarrow_{5,\sigma} \partial R_N). \end{aligned}$$

□

2, 3–Arm Exponents in Half Space. Similarly elementary arguments also yield *integer* exponents for 2, 3–arm exponents in the *half space*, i.e., the events

$$B_{2,\sigma}(0,N), \quad B_{3,\sigma}(0,N)$$

refer to connections from 0 to ∂R_N (centered at 0) restricted to $R_N \cap \{y \geq 0\}$. (In particular, in this setting e.g., BY is *not* identified with YB.)

Lemma. Suppose $p = p_c$. Let $\sigma = BY$, then as $N \to \infty$,

$$B_{2,\sigma}(0,N) \sim N^{-1}.$$

Proof. The argument proceeds much like in the 5–arm case. We have here that e.g., that for all $v \in [-N/4, N/4]$,

$$\mathbb{P}(v \rightsquigarrow_{2,\sigma} \partial R_N) \sim \mathbb{P}(0 \rightsquigarrow_{2,\sigma} \partial R_N); \quad \text{(all restricted to } \{y \geq 0\}).$$

Now for (any) prescribed landing intervals I define the event

$$B_v = \{v \rightsquigarrow^{I}_{2,\sigma} \partial R_N\}.$$

As an *upper bound* we then have the estimate that

$$1 \geq \mathbb{P}(\cup_{v\in[-N/4,N/4]} B_v) = \sum_{v\in[-N/4,N/4]} \mathbb{P}(B_v) \sim N \cdot \mathbb{P}(0 \rightsquigarrow_{2,\sigma} \partial R_N).$$

Indeed, it is easy to see that with *fixed* landing intervals I_b, I_y for the *blue* and *yellow* arms (say ordered *clockwise*) it is not possible to have B_v and B_w to occur together, for $v \neq w$ since e.g., one of the required arms for B_w is guaranteed to be *not possible* because of the occurrence of B_v.

[picture of half space 2–arm event with $-N/2, N/2, I_b, I_y$ labeled...]

The lower bound again involves RSW constructions: we can construct a *blue* crossing from $[-N/4, 0]$ to I_b and a *yellow* crossing from $[0, N/4]$ to I_y, then B_v occurs for the site v which is the *rightmost* site on $[-N/4, N/4]$ which is connected to I_b, since again the *absence* of a yellow crossing to ∂R_N would contradict the *rightmostness* of v. We claim that this yellow crossing can be joined to the original yellow crossing to I_y: in fact, we may *envision* a *blue–yellow interface* going from v to the portion of ∂R_N *between* I_b *and* I_y which separates

- the *largest blue cluster* connected to the portion of ∂R_N starting from v and going clockwise until it subsumes I_b (the "blue" boundary) from
- the *largest yellow cluster* connected to the portion of ∂R_N starting from v and going *counterclockwise* until it subsumes I_y (the "yellow" boundary);

This yellow crossing is then part of the *yellow connected cluster* connected to the "yellow" boundary and *topologically* it must be to *left* of the *original yellow crossing* from $[0, N/4]$ to y_b constructed via RSW. In any case, the two yellow crossings can be joined by a crossing whose probability is bounded *below* by the probability of a *left right crossing* of $R_{N/2}$.

[picture of RSW construction with $-N/2, N/2, I_b, I_y$ labeled and relevant portions of ∂R_N colored for envisionment of an interface...]

□

Lemma. Suppose $p = p_c$. Let $\sigma = BYB$, then as $N \to \infty$,

$$B_{3,\sigma}(0, N) \sim N^{-2}.$$

Proof. Let us denote some (any) prescribed landing areas by I_{b_1}, I_y, I_{b_2}. For the *lower bound*, we may use RSW to construct a *blue crossing* from I_{b_1} to I_{b_2} *going through* $R_{N/2} \cap \{y \geq 0\}$ and look at the crossing *closest* (in Euclidean distance) to I_y, so that all sites *on the crossing* has a *yellow* connection to I_y.

[picture of 3–arm half space event, with I_{b_1}, I_y, I_{b_2}, N/2, etc., labeled...]

For the *upper bound* we first note that for all $v \in R_{N/2}$,

$$\mathbb{P}(v \rightsquigarrow_{3,\sigma} \partial R_N) \sim \mathbb{P}(0 \rightsquigarrow_{3,\sigma} \partial R_N); \quad \text{(all restricted to } \{y \geq 0\}).$$

Denoting by B_v the relevant event, we will arrive at the *upper bound*

$$1 \geq \mathbb{P}(\cup_{v \in R_{N/2}} B_v) \sim N^2 \cdot \mathbb{P}(0 \rightsquigarrow_{3,\sigma} \partial R_N).$$

Here to arrive at the last $\sim$ we interpret $B_{3,\sigma}$ as the *derivative* of a *blue crossing event*: the event B_v implies that there is a *blue crossing* from I_{b_1} to I_{b_2} for v but *not* for e.g., the *site below* v.

For this purpose we will also consider an *auxiliary crossing event*: let define the events (all restricted to the *upper half plane*)

$$C = \{\text{there exists a blue crossing from } I_{b_1} \text{ to } I_{b_2} \text{ going through } R_{N/2}\}$$

and

$$C' = \{\text{there exists a blue crossing from } I_{b_1} \text{ to } I_{b_2} \text{ going through the } \textit{upper half of } R_{N/2}\}.$$

[picture of both C and C' with box divided in half by dash...]

By RSW both events are of *order unity* and clearly, $\mathbb{P}(C) > \mathbb{P}(C')$ and so

$$\mathbb{P}(C) - \mathbb{P}(C') \lesssim 1.$$

Next we define *defective domain crossing event*: *enumerate* sites in the *lower half* of $R_{N/2}$ *consequtively* $v_1, \ldots, v_K$ (with $K \sim N^2$) and define the *decreasing domains*

$$R_j = R_{N/2} \setminus v_1, \ldots, v_j$$

and the *associated crossing events*

$$C \supseteq C_j = \{\text{there exists a blue crossing from } I_{b_1} \text{ to } I_{b_2} \text{ going through } R_j\}.$$

It is then clear that

$$C = C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots \supseteq C_K = C'$$

and $B_{v_j} \supseteq C_j \setminus C_{j+1}$ but with *comparable probability* by RSW:

$$\mathbb{P}(B_{v_j}) \sim \mathbb{P}(C_j \setminus C_{j+1}).$$

[picture of site distortion with 3–arm corresponding to "derivative"... grid bottom half of $R_{N/2}$...]

Therefore, we may write a *telescoping estimate*

$$\begin{aligned} 1 &\gtrsim \mathbb{P}(C) - \mathbb{P}(C') \\ &= \sum_j \mathbb{P}(C_{j+1}) - \mathbb{P}(C_j) \\ &= \sum_j \mathbb{P}(C_{j+1} \setminus C_j) \\ &\sim \sum_j \mathbb{P}(B(v_j)) \\ &\sim N^2 \cdot \mathbb{P}(0 \rightsquigarrow_{3,\sigma} \partial R_N). \end{aligned}$$

□

Remark. It is not possible to so easily write such a *telescoping estimate* without some *monotonicity*, since usually we have that

$$\mathbb{P}(A) - \mathbb{P}(B) = \mathbb{P}(A \setminus B) - \mathbb{P}(B \setminus A).$$

Remark. We do envision the results here to be *asymptotic* ($N \to \infty$) results but for p *near* p_c the proofs are only valid *for* $N \leq L(p)$. However, there is no "contradiction" as it is the case that $L(p) \to \infty$ as $p \to p_c$.

References.

1. *Near Critical Percolation in Two Dimensions* by Pierre Nolin. Electronic Journal of Probability, Vol. 13, no. **55**, 1562–1623 (2008).

2. (See Lemma 3.8.) *On Convergence to SLE6 I: Conformal Invariance for Certain Models of the Bond-Triangular Type* by I. Binder, L. Chayes, H. K. Lei. Journal of Statistical Physics, Vol. 141, Iss. **2**, 359–390 (2010).