V. "Fences and Corridors"

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Kesten's "Fences". Here we introduce multiple crossing events of an annulus and quantify notions of "well–separatedness". These technical results will be used in establishing the *scaling relations* (algebraic equations relating various *critical exponents*).

Definition. Let $0 < n < N$ and $j > 0$ be integers. Also let

$$\sigma = (\sigma_1, \ldots, \sigma_j) \in 2^{[j]}$$

denote a sequence of *colors*, e.g., (blue, yellow, yellow, blue) and we identify σ up to *cyclic permutations*. We define the *arm crossing* events to be

$$A_{j,\sigma}(n, N) := \{\omega : \partial B_n \rightsquigarrow_{j,\sigma} \partial B_N\},$$

i.e., the event of j *disjoint monochromatic* crossings of the annulus $B_N \setminus B_n$, with the colors (up to cyclic order) prescribed by σ. We will sometimes omit (n, N) when it is not needed.

[picture of example of arm event...]

Remark. By *Reimer's inequality* we immediately obtain the *multiplicative* bound that

$$\mathbb{P}(A_{j+j',\sigma\sigma'}) \leq \mathbb{P}(A_{j,\sigma}) \cdot \mathbb{P}(A_{j',\sigma'}).$$

Next we define what it means for crossings to be "well–separated".

Definition. Consider the box $B_M = [0, M] \times [0, M]$ and denote by

$$\mathcal{C} = \{c_i, \sigma_i\}_{i=1}^{j}$$

a sequence of (*deterministic*) *left right* crossings with colors encoded by σ. Given a crossing c_i, let

$$z_i = \text{landing point on the } \textit{right} \text{ side of } B_M.$$

Also, let $\eta > 0$, and let

$$r_i = z_i + [0, \sqrt{\eta}M] \times [-\eta M, \eta M]$$

be small rectangles attached to each z_i. (This will be used to quantify *extension* of crossings.)

Then we say that $\mathcal{C}$ *is well–separated at scale* η if:

- z_i's are far from each other and from the corners on the *right*:

$$\forall i \neq j, \quad \text{dist}(z_i, z_j) \geq 2\sqrt{\eta}M$$

$$\forall i, \quad \text{dist}(z_i, (0, M)) \geq 2\sqrt{\eta}M, \quad \text{dist}(z_i, (M, M)) \geq 2\sqrt{\eta}M.$$

[picture of well–separated with space between crossings and scales labeled...]

- Each r_i is crossed *vertically* by some crossing $\tilde{c}_i$ of r_i and

$$c_i \rightsquigarrow \tilde{c}_i \quad \text{in } B_{\sqrt{\eta}M}(z_i).$$

  Here all crossings are of the same color: we are requiring a small extension of c_i.

[picture of only right side of B_M and off scale local picture of what happens at z_i, with the various crossings labeled...]

Finally, we say that $\mathcal{C}$ can be *made well–separated* if there exists a well–separated $\mathcal{C}'$ with the same endpoint as $\mathcal{C}$ *on the left*.

[picture of $\mathcal{C}$, not well–separated in solid, a well–separated $\mathcal{C}'$ in dots...]

This notion can be directly adapted to annular crossing, where we separately require well–separatedness on all relevant sides used by the crossings.

Definition. Let $0 < n < N$ be integers and let $\eta, \eta' > 0$. Denote by

$$A_{j,\sigma}^{\eta;\eta'}(n, N) := \{\partial B_n \rightsquigarrow_{j,\sigma}^{\eta;\eta'} \partial B_N\}$$

the *subevent* of $A_{j,\sigma}$ with the additional stipulation that we have well–separatedness on all relevant sides.

[picture of annulus with well–separatedness imposed so that corners, etc., are shaded out: for the internal box corner avoidance means avoiding a region *surrounding* each corner and the rectangles r_i lies *inside* the small box...]

The definition can be generalized to *landing intervals*, i.e., intervals I_i such that $z_i \in I_i$:

Definition. Consider *landing intervals* $\{I_i\}_{i=1,\ldots,j}$ on ∂B_N. We say they are *η–separated* if

- $\operatorname{dist}(I_i, I_{i+1}) \geq 2\sqrt{\eta}N$
- $\operatorname{dist}(I_i, (N, 0)) \geq 2\sqrt{\eta}N, \quad \operatorname{dist}(I_i, (N, N)) \geq 2\sqrt{\eta}N.$

Also, we say they are landing intervals of *scale* η if

$$\text{length}(I_i) \geq \eta N.$$

We identify landing intervals up to rescaling: $I \sim \tilde{I}$ if $I \in \partial S_n, \tilde{I} \in \partial S_N$, with $\tilde{I} = \frac{N}{n} \cdot I$.

We can now define the events

$$A_{j,\sigma}^{\eta;\eta'}(n, N) \supseteq A_{j,\sigma}^{\eta,I;\eta',I'}(n, N) := \{\partial B_n \leadsto_{j,\sigma}^{\eta,I;\eta',I'} \partial B_N\},$$

so that

$$z_i \in I_i, \quad z_i' \in I_i'.$$

Finally, we define the *relaxed* events

$$A_{j,\sigma}^{I;I'}(n, N) := \{\partial B_n \leadsto_{j,\sigma}^{I;I'} \partial B_N\},$$

with the requirement of η–separatedness replaced by (merely) *disjointness* and *without* requirement of *free space* (i.e., the r_i's).

Remark. Note that we have the containments

$$A_{j,\sigma}^{\eta,I;\eta',I'} \quad \subseteq \quad A_{j,\sigma}^{\eta;\eta'} \quad \subseteq A_{j,\sigma};$$

$$A_{j,\sigma}^{\eta,I;\eta',I'} \quad \subseteq \quad A_{j,\sigma}^{I;I'} \quad \subseteq A_{j,\sigma}.$$

Characteristic Length Revisited. Recall that

$$p > p_c \quad \iff \quad \liminf_{L\to\infty} C_p(L, L) = 1 \quad \iff \quad \limsup_{L\to\infty} C_{p^*}^*(L, L) = 0 \quad \iff \quad p_* < p_c^*,$$

whereas at $p = p_c$, there are non–trivial upper and lower bounds for crossing probability *at all scales*: there exists $0 < \sigma, \sigma' < 1$ such that

$$1 - \sigma' \leq C_{p_c}(L, L) \leq \sigma.$$

For p *off criticality* we can define a *characteristic length* up to which the *crossing probability* is well–behaved: given $\varepsilon_0 > 0$, let

$$L_{\varepsilon_0}(p) = \inf_n \{C_p(n,n) \leq \varepsilon_0\}, \quad p < p_c;$$

$$L_{\varepsilon_0}(p) = \inf_n \{C_p^*(n,n) \leq \varepsilon_0\}, \quad p > p_c.$$

so that if e.g., $p < p_c$, then

$$C_p(L_{\varepsilon_0}(p)-1, L_{\varepsilon_0}(p)-1) \geq \varepsilon_0, \quad C_{p^*}(L_{\varepsilon_0}(p)-1, L_{\varepsilon_0}(p)-1) \leq 1-\varepsilon_0$$

and

$$C_p(L_{\varepsilon_0}(p), L_{\varepsilon_0}(p)) \leq \varepsilon_0 \quad \Longrightarrow \quad C_{p_*}^*(L_{\varepsilon_0}(p), L_{\varepsilon_0}(p)) \geq 1-\varepsilon_0.$$

We make the following observations:

- Thus, for *all* $n < L_{\varepsilon_0}$, both the *direct* and *dual* crossing probabilities are bounded away from 0 and 1 and thus we have *scale invariance* up to L_{ε_0} and may use RSW constructions.
- If $p_c = 1/2$ (particularly *site percolation* on the *triangular lattice*) then by *complete symmetry* we have $L_{\varepsilon_0}(p) = L_{\varepsilon_0}(1-p)$.
- We also observe that if $\varepsilon_0 < \sigma$ (lower bound for crossing probability at criticality) then

$$L_{\varepsilon_0} \to \infty \quad \text{as} \quad p \nearrow p_c,$$

since otherwise we have that $C_{p_n}(L_{\varepsilon_0}(p_n), L_{\varepsilon_0}(p_n)) \leq \varepsilon_0 < \sigma$ which as $p_n \nearrow p_c$ *contradicts* the fact that $C_{p_c}(L, L) < \sigma$ for all L. A similar argument with the *dual model* when $\varepsilon_0 < 1 - \sigma'$ gives that also,

$$L_{\varepsilon_0} \to \infty \quad \text{as} \quad p \searrow p_c.$$

- Therefore, for

$$\varepsilon_0 < \min\{1-\sigma', \sigma\}$$

the characteristic length *diverges at* $p = p_c$; we take this to be the convention and 1) will now suppress the ε_0 subscript and 2) in what follows *if* p is *off critical* then *all relevant length scales* are assumed to be less than $L(p)$.

Well–Separatedness. We will first establish the following:

Theorem. *Fix some* $0 < p < 1$ *and* $N > n$ *positive integers. Let* $j \geq 1$ *be an integer,* σ *be some color sequence, and* $\eta_0, \eta'_0 \in (0,1)$. *Then*

$$\mathbb{P}_p(A^{\tilde{\eta};\eta'}_{j,\sigma}(n,N)) \quad \sim_{\eta_0,\eta'_0} \quad \mathbb{P}_p(A_{j,\sigma}(n,N)),$$

uniformly for all $\tilde{\eta}, \eta' \geq \eta_0, \eta'_0$. *Here* $\sim$ *means upper and lower bound up to a constant.*

Remark. We will later obtain a *strengthening* of this theorem which also allows for prescription of *landing areas*. We do this in two stages in order to make more explicit the "corridors" construction.

We first state some consequences of the RSW estimates for the "decorated" $A_{j,\sigma}$ events.

Remark. In what follows we will only consider the *external* boundary (i.e., landings on the outer boundary of the annulus) since arguments for the *internal* boundary is similar and will *suppress the indices* η', I' whenever possible (with the implicit understanding that it is possible to take a *different scale* for the *internal* boundary).

Proposition. For fixed $j \geq 1$ and color sequence σ, uniformly for all $\eta \geq \eta_0$:

1. Extendability:

$$\mathbb{P}(A^{\eta_0}_{j,\sigma}(n,2N)) \sim \mathbb{P}(A^{\eta}_{j,\sigma}(n,N))$$

$$\mathbb{P}(A^{\eta}_{j,\sigma}(n/2,N)) \sim \mathbb{P}(A^{\eta_0}_{j,\sigma}(n,N))$$

2. Quasi–multiplicativity: (here $\bullet$ denotes *absence* of condition)

$$\mathbb{P}(A^{\bullet;\eta}_{j,\sigma}(n_1,n_2/4)) \cdot \mathbb{P}(A^{\eta';\bullet}_{j,\sigma}(n_2,n_3)) \sim \mathbb{P}(A_{j,\sigma}(n_1,n_3)).$$

3. For $\eta > 0$ *there exist intervals* I of size η such that

$$\mathbb{P}(A^{\eta,I}_{j,\sigma}(n,N)) \gtrsim_\eta \mathbb{P}(A^{\eta}_{j,\sigma}(n,N)).$$

4. There exist *exponents*

$$0 < \alpha_j, \alpha < \infty$$

such that

$$\left(\frac{n}{N}\right)^{\alpha_j} \lesssim_j \mathbb{P}(A^{\eta}_{j,\sigma}(n,N)) \lesssim \left(\frac{n}{N}\right)^{\alpha}.$$

Proof. These are fairly straightforward RSW/FKG "gluing" constructions like before. Indeed, the definitions of fences etc., is precisely to permit the appropriate continuations.

Even though the arm colors may be different and so we are considering intersections of increasing *and* decreasing events, because of well–separatedness, *independence* can be used to show that

$$\mathbb{P}(\tilde{A}^+ \cap \tilde{A}^- \mid A^+ \cap A^-) \geq \mathbb{P}(\tilde{A}^+) \cdot \mathbb{P}(\tilde{A}^-),$$

where A^+ and A^- depend on *disjoint sets of vertices* and $\tilde{A}^\pm$ may in addition depend on another (*disjoint* from the previous) set of indices.

1. Here we may realize $A(n, 2N)$ as a continuation of $A(n,N)$ "one arm at a time". E.g., focusing on one arm landing on the right side, we have the inclusion

$$A^{\eta}_{j,\sigma}(n,N) \cap \mathcal{C}(N + \sqrt{\eta_0}N, 2\eta_0 N) \cap \mathcal{C}(\sqrt{\eta_0}N, 2\eta_0 N) \subseteq A^{\eta_0}_{j,\sigma}(n, 2N),$$

so that estimating the *crossing probabilities* using RSW, we indeed have

$$\mathbb{P}(A^{\eta}_{j,\sigma}(n,N)) \lesssim_{\eta_0} \mathbb{P}(A^{\eta_0}_{j,\sigma}(n,2N)).$$

[picture of continuation from ∂B_N to ∂B_{2N}...]

The reverse inequality is of course entirely clear.

2. This can be done in a similar fashion.

3. Here we note that e.g., the outer boundaries of the annuli can be covered with at most $8\eta^{-1}$ intervals I of length η, and we may *partition* the event $A^{\eta}_{j,\sigma}$ according to the landing points z_i, and therefore *at least one* I contains at least an *average* number of landing points, i.e.,
$$\frac{\mathbb{P}(A^{\eta}_{j,\sigma})}{8\eta^{-1}} \leq \mathbb{P}(A^{\eta,I}_{j,\sigma}).$$

4. For the *upper bound* we note that the crossing event can be avoided by *circuits* in annuli: assume without loss of generality that σ requires at least one *yellow* crossing. We may set up of the order $\log(N/n)$ annuli (all of the same *aspect ratio*) inside $B_N \backslash B_n$, for each of which with independent *independent* probability
$$\alpha' = \mathbb{P}(\exists \text{ blue circuit}) > 0$$
there is a blue circuit preventing the relevant arm crossing. Therefore
$$\mathbb{P}(A^{\eta}_{j,\sigma}(n,N)) \lesssim (1-\alpha')^{\log(N/n)} \lesssim (e^{-\alpha'})^{\log(N/n)} = \left(\frac{n}{N}\right)^{\alpha'}.$$

[picture of logarithmically many annuli, with some circuits...]

For the *lower bound* we iterate item 1 of the order $\log(N/n)$ times (e.g., consider a "coarsened" scale defined by n and double each time) each time except the first incurring
$$\lambda_j = e^{-\alpha_j} = \text{cost of continuing the crossing.}$$

The intersection of all such events lies in $A^{\eta}_{j,\sigma}$ and so with e.g., $C_j = \mathbb{P}(A(n, 2n))$ the probability at the initial scale, we have

$$C_j \cdot (e^{-\alpha_j})^{\log(N/n)} = C_j \left(\frac{n}{N}\right)^{\alpha_j} \lesssim \mathbb{P}(A^{\eta}_{j,\sigma}).$$

$\square$

Once the theorem is proved, we will then have the above results for the *unadorned* arm crossing events $A_{j,\sigma}$ (up to more constants).

Proof of Well–Separatedness. It is certainly clear that

$$\mathbb{P}(A^{\eta}_{j,\sigma}) \leq \mathbb{P}(A_{j,\sigma}),$$

so it is sufficient to establish the reverse inequality (up to a constant). Also, it is sufficient to consider n, N such that

$$n = 2^k, N = 2^K, \quad k \in \mathbb{N}.$$

(For generic n, we find k, K such that $2^{k-1} < n \leq 2^k, 2^K \leq N < 2^{K+1}$.)

We consider some *fixed scale* of separation $\tilde{\eta} \geq \eta_0$ and aim to show that

$$\mathbb{P}(A^{\tilde{\eta}}_{j,\sigma}(2^k, 2^K)) \gtrsim_{\eta_0} \mathbb{P}(A_{j,\sigma}(2^k, 2^K)).$$

From now on we fix j, σ and suppress these indices.

The proof consist of these steps:

- First consider *doubling of a single scale*. Observe that each annulus can be decomposed into *four U–shaped regions* such that the existence of an arm in the annulus implies the existence of a *shortway crossing* of the U–region.

[annulus decomposed into four U–shaped regions; shade portions of the ∂U on which landings should be made well–separated, different for internal/external boundaries...]

- For both *external* and *internal* boundaries of the annulus, *given* $\delta > 0$, show that *there exists* $\eta(\delta) > 0$ such that the crossings can be made $\eta(\delta)$*–well–separated* with probability in excess of $1 - \delta > 0$.

- Sum up over $K - k$ subannuli: we consider the *external boundary* (the case of the internal boundary is done similarly, summing in the "other direction").

[picture of $A(2^k, 2^K)$ divided into "even" log scales, summing from large to small...]

Let

$$\mathcal{U}_{2^\ell}(\eta) = \{\text{crossing of } \textit{all } U\text{–shaped region corresponding to } A(2^{\ell-1}, 2^\ell) \text{ is } \eta\text{–well–separated}\}.$$

Then we may *decompose* the event $A(2^k, 2^K)$ according to whether the crossings are *well–separated*:

$$A(2^k, 2^K) \subseteq A^{\bullet;\eta}(2^k, 2^K) \cup [\ [\mathcal{U}_{2^K}(\eta)]^c \cap A(2^k, 2^{K-1})\ ].$$

From the previous item, we have that for all $\ell > 0$,

$$\mathbb{P}([\mathcal{U}_{2^\ell}(\eta)]^c) \le 4\delta,$$

and, also, the events $\mathcal{U}_{2^{K-1}}(\eta)$ and $A(2^k, 2^{K-1})$ occur in *disjoint* regions and are *independent*. Therefore,

$$\mathbb{P}(A(2^k, 2^K) \le \mathbb{P}(A^{\bullet;\eta}(2^k, 2^K)) + 4\delta \cdot \mathbb{P}(A(2^k, 2^{K-1})).$$

Iterating $K-k$ times, we obtain the estimate

$$\mathbb{P}(A(2^k,2^K)) \le \mathbb{P}(A^{\bullet;\eta}(2^k,2^K)) + 4\delta\cdot\mathbb{P}(A^{\bullet;\eta}(2^k,2^{K-1})) + (4\delta)^2\cdot\mathbb{P}(A^{\bullet;\eta}(2^k,2^{K-2})) + \dots$$
$$+ (4\delta)^{K-k-1}\cdot\mathbb{P}(A^{\bullet;\eta}(2^k,2^{k+1})) + (4\delta)^{K-k}.$$

- To continue note that

  1) from the previous Proposition, 1, we have for any m,

  $$\mathbb{P}(A^{\bullet;\eta}(2^m,2^{m+1}) \lesssim_\eta \mathbb{P}(A^{\bullet;\eta_0}(2^{m-1},2^m),$$

  that is, we can go from η–*separated* to η_0–*separated* at a constant cost;

  2) going from $2m$ to $4m$ *remaining at scale* η_0 incurs some cost C_0. That is, we have

  $$\mathbb{P}(A^{\bullet;\eta}(2^k,2^{K-\ell}) \lesssim_\eta C_0^{\ell-1}\cdot\mathbb{P}(A^{\bullet;\eta_0}(2^k,2^K)).$$

  The previous estimate becomes then (we may drop the term involving $A^{\bullet;\eta}(2^k,2^K)$ at the cost of an upper bound; i.e., "shorten" the crossing):

  $$\mathbb{P}(A(2^k,2^K)) \lesssim_\eta (1+4\delta C_0+\dots+(4\delta C_0^{K-k-1})\cdot\mathbb{P}(A^{\bullet;\eta_0}(2^k,2^K)).$$

  We may conclude by *choosing* δ so that

  $$4\delta C_0 < 1;$$

  3) finally, using the previous Proposition, 1, we may return to the original scale $\tilde{\eta}$:

  $$\mathbb{P}(A^{\bullet;\eta_0}(2^k,2^K) \lesssim_{\eta_0} \mathbb{P}(A^{\bullet;\tilde{\eta}}(2^k,2^K).$$

It remains to prove:

Lemma. *Given $\delta > 0$, there exists $\eta(\delta) > 0$ such that for all $\eta \leq \eta(\delta)$,*

$$\mathbb{P}(\mathcal{U}_N(\eta)) \geq 1 - \delta.$$

Proof. (Sketch) The statement follows from a sequence of RSW constructions: The event of a set of crossing being not η–well–separated can be *prevented* by existence of suitable circuits in annuli.

1. (Corners) Near the corners, setting up of the order $-\log \eta$ annuli we have that with *uniform* and *independent* probability $> \delta'$ there exists blue and/or yellow circuits, preventing the possibility of a crossing going near said corner:

   [picture annuli around corners with circuits of both colors...]

   Therefore,

   $$\mathbb{P}(\exists \text{ a crossing landing near the corners}) \lesssim (\delta_1)^{-C \log \eta},$$

   for some constant $C > 0$.

2. Since the existence of a circuit in the *original* annuli prevents the possibility of a crossing in U_N,

   $$\mathbb{P}(\exists\ 1 \text{ monochromatic crossing in } U_N) \leq (1 - \delta_2),$$

   for some $\delta_2 > 0$. Thus, by *Reimer's inequality*,

   $$\mathbb{P}(\exists \text{ at least } H \textit{ disjoint} \text{ monochromatic crossings in } U_N) \leq (1 - \delta_2)^H.$$

   We therefore choose T to be such that

   $$(1 - \delta_2)^T < \delta/2,$$

so that it is *unlikely* for there to be more than T crossings. Notice that T is *fixed* (in particular *independent of* η).

3. To handle the crossings themselves, first *enumerate* crossings from the bottom (look at the *lowest monochromatic crossing*, then the next, etc.) and call the resulting set $\mathcal{C}$. *Conditioned* on the k^{th} lowest such crossing γ, the region above is *independent*, and thus we can perform another RSW construction as in the corners case to ensure no other crossing comes within $\sqrt{\eta}$ of γ.

   Yet another RSW construction gives the appropriate *extension* of any such crossing. We say γ is *good* if all such circuits exist, so then we have

$$\mathbb{P}(\gamma \text{ is } \textit{not} \text{ good }) \lesssim (\delta_3)^{-C' \log \eta},$$

   for some $\delta_3 > 0$.

   [picture annuli above right landing point of γ preventing other crossings and continuation...]

4. Therefore with *high probability* the set $\mathcal{C}$ can be made well–separated:

$$\begin{aligned}
\mathbb{P}(\mathcal{C} \textit{ not } \eta\text{–well–separated}) &\leq \mathbb{P}(|\mathcal{C}| \geq T) \\
&\quad + \sum_{k=1}^{T-1} \mathbb{P}(|\mathcal{C}| \geq k \text{ and } \gamma_k \text{ is } \textit{not} \text{ good}) \\
&\quad + \mathbb{P}(\exists \text{ crossing landing near the corners}) \\
&\lesssim \delta/2 + (T-1) \cdot (\delta_3)^{-C' \log \eta} + (\delta_1)^{-C \log \eta} \\
&< \delta,
\end{aligned}$$

for $\eta > 0$ sufficiently small.

5. Finally, we observe that it is sufficient to ensure $\mathcal{C}$ is well–separated. Indeed, by its construction, $\mathcal{C}$ is *maximal* amongst all *disjoint* sets of monochromatic crossings in U_N, so given some $\gamma' \in \mathcal{C}'$ some other set of crossings:

   (a) γ' must *intersect* some $\gamma \in \mathcal{C}$ and further

   (b) taking γ to be the *lowest* such curve, we see that γ' hits γ from *above* and cannot go *below* γ.

   We can therefore consider instead the curve $\tilde{\gamma}$ which has *first portion equal to* γ' and the *very last* portion equal to γ so that $\tilde{\gamma}$ can be extended via the extension of γ. (This argument shows exactly that $\mathcal{C}'$ can be *made* well–separated.)

[picture of γ' hitting γ from above either before or after landing of continuation...]

□

"Corridors" and Rearrangements. As mentioned before, a consequence of existence of *fences* is that arms can also be *rearranged* at constant cost. For simplicity we will explicitly consider $A_{2,BY}$.

Corollary. Fix some $0 < p < 1$. For any integers $N \gg n \geq 0$ define

$$A_{2,BY} \supseteq \Lambda_{2,BY} = \{\text{there are landing areas } I_0 \text{ of scale } \eta_0 \text{ such that there is a } \textit{yellow} \text{ crossing landing on the } \textit{top boundary} \text{ and a } \textit{blue} \text{ crossing landing on the } \textit{left boundary}\}.$$

[picture of two arms with correct landing locations...]

Then for $\eta_0 > 0$ sufficiently small (e.g.,

$$\mathbb{P}(A_{2,BY}(n,N)) \sim_{\eta_0} \mathbb{P}(\Lambda_{2,BY}(n,N)).$$

Proof. As in the proof of the theorem, it suffices to consider $n = 2^k, N = 2^K$ for $K > k \geq 0$. From the previous Proposition on the *adorned crossing events*, 3, we already have

$$\mathbb{P}(A^{\bullet;\eta_0}(2^k, 2^K)) \lesssim_{\eta_0} \mathbb{P}(A^{\bullet;\eta_0,I}(2^k, 2^K))$$

for some landing interval I. Now we will show that for any $m \geq 0$,

$$\mathbb{P}(A^{\bullet;\eta,I}(2^{m-1}, 2^m)) \lesssim_{\eta_0} \mathbb{P}(A^{\bullet;\eta_0,I_0}(2^m, 2^{m+1})).$$

Indeed, going from $A(2^{m-1}, 2^m)$ to $A(2^m, 2^{m+1})$ we can by RSW relocate the arms inside a U–shaped region at the cost of an η_0–dependent constant (since the crossings stay away from corners and are well–separated, the appropriate η_0–*corridors* can be constructed):

[picture of two arms both landing on right boundary in inner square, with blue arm above yellow and corridors relocating blue arm to top and yellow arm continued to right...]

In this particular case the argument shows that we may upper bound the left hand side by crossings with yellow landing on the right boundary and blue to the top:

$$A^{\bullet;\eta,I}(2^{m-1}, 2^m) \cap \mathcal{C}(2^m + \sqrt{\eta_0} \cdot 2^m) \cap \mathcal{C}(2\eta_0 \cdot 2^m, \sqrt{\eta_0} \cdot 2^m) \cap \mathcal{C}(\eta_0 \cdot 2^m, \lambda \cdot 2^m)) \subseteq A^{\bullet;\eta,I}(2^m, 2^{m+1}).$$

(Here the term involving λ bounds the crossing in the *corridor to the top*, so $\lambda \leq 3/2$.) On the next scale, another corridors construction yields arms landing on the *prescribed sides* of the square (top and left):

[picture of blue arm to top, yellow arm to bottom, relocated to correct location on next largest scale...]

We may now finish by plugging this into the remaining steps of the proof of the theorem (sum over scales). In other cases, e.g., if we started with both the yellow and blue crossings to the top, we may relocated the *blue* arm going *counterclockwise*:

[picture of blue and yellow arm to top, blue to right of yellow, arrow indicating blue arm relocated clockwise...]

□

Corollary. Fix some $0 < p < 1$ and integers $N \gg n \geq 0$. Let $j \geq 1$ be an integer, σ be some color sequence, and $\eta_0, \eta_0' \in (0,1)$. Then

$$\mathbb{P}_p(A_{j,\sigma}^{\eta,I;\eta',I'}(n,N)) \quad \sim_{\eta_0,\eta_0'} \quad \mathbb{P}_p(A_{j,\sigma}(n,N)),$$

uniformly for all $\eta, \eta' \geq \eta_0, \eta_0'$. Here $\sim$ means upper and lower bound up to a constant.

Proof. This can be done as in the previous corollary: successively relocate arms and the theorem (and more careful corridors constructions) yield landings in the prescribed areas. □

Consequences. We finish by listing some consequences which will be useful for us.

Corollary. Fix some $j \geq 1$, color sequence σ and $0 < p < 1$. Then for $N > n > 0$, $n_3 > n_2 > n_1 > 0$,

1. $\mathbb{P}(A_{j,\sigma}(n, 2N)), \mathbb{P}(A_{j,\sigma}(n/2, N) \sim \mathbb{P}(A_{j,\sigma}(n, N);$
2. $\mathbb{P}(A_{j,\sigma}(n_1, n_2) \cdot \mathbb{P}(A_{j,\sigma}(n_2, n_3)) \sim \mathbb{P}(A_{j,\sigma}(n_1, n_3));$
3. for any $x \in R_{N/2}$,
$$\mathbb{P}(x \rightsquigarrow \partial R_N) \sim \mathbb{P}(0 \rightsquigarrow \partial R_N).$$

Proof. The first two items follow from the corresponding properties for the *adorned crossing events*. The last item follows from the first one, since by *translation invariance* (consider the *shifted* boxes centered at x) we have the inequalities

$$\mathbb{P}(0 \rightsquigarrow \partial R_{N/2}) \leq \mathbb{P}(x \rightsquigarrow \partial R_N) \leq \mathbb{P}(0 \rightsquigarrow \partial R_{2N}).$$

[annulus and its shift...]

$\square$

References.

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