FK Representation/Random Cluster Expansion

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Let's do the simplest case of the Ising model on the square lattice with zero external field on some (finite) domain Λ. Let's forget boundary conditions for now. Let $\sigma \in \{+1, -1\}^{|\Lambda|}$ be a spin configuration. The Hamiltonian is

$$-\beta\mathcal{H} = \sum_{\langle x,y \rangle} J\delta_{\sigma_x \sigma_y},$$

where the sum over $\langle x, y\rangle$ is over all nearest neighbor sites in Λ. We will drop the subscript Λ from now on. The partition function is

$$Z_\Lambda \equiv \sum_\sigma e^{-\beta\mathcal{H}} = \sum_\sigma e^{\sum_{\langle x,y\rangle} J\delta_{\sigma_x\sigma_y}},$$

where the sum over σ_Λ is over all possible spin configurations in Λ.

We will expand the inner sum with the goal of ending up with some semblance of the random cluster weights: Let ω be a bond configuration (of open and closed bonds) and let $p \in [0, 1]$, then

$$W(\omega) \propto p^{|\omega|}(1-p)^{E-|\omega|}2^{c(\omega)},$$

where $|\omega|$ is the number of occupied bonds, E is the total number of bonds and $c(\omega)$ is the total number of connected components of ω, counting isolated sites.

Let $S = e^J - 1$, then we have

$$\begin{aligned}
Z_{[\sigma]} \equiv Z_\Lambda &= \sum_\sigma e^{\sum_{\langle x,y\rangle} J\delta_{\sigma_x\sigma_y}} \\
&= \sum_\sigma \prod_{\langle x,y\rangle} e^{J\delta_{\sigma_x\sigma_y}} \\
&= \sum_\sigma \prod_{\langle x,y\rangle} (e^J - 1)\delta_{\sigma_x,\sigma_y} + 1 \\
&= \sum_\sigma \prod_{\langle x,y\rangle} (S\delta_{\sigma_x\sigma_y} + 1)
\end{aligned}$$

where to obtain the third equality, consider separately the cases where $\sigma_x = \sigma_y$ and $\sigma_x \neq \sigma_y$. In the last product, we get $1 + S$ if $\sigma_x = \sigma_y$ and 1 otherwise, so letting

$$I(\sigma) = \#\{\langle x, y\rangle \mid \sigma_x = \sigma_y\},$$

we have (continuing the above expressions)

$$\begin{aligned} Z_{[\sigma]} &= \sum_\sigma (S+1)^{I(\sigma)} \\ &= \sum_\sigma \sum_{k=0}^{I(\sigma)} \binom{n}{k} S^k 1^{I(\sigma)-k}. \end{aligned}$$

Now we begin the identification with bond configurations. Each term in the inner sum above corresponds to a subset of $\omega(\sigma)$, where

$$\omega(\sigma) = \{\langle x, y\rangle \mid \sigma_x = \sigma_y\}$$

is the set of bonds in Λ such that the spins (according to σ) of the endpoints agree. So we can continue as follows

$$Z_{[\sigma]} = \sum_\sigma \sum_{\omega \subset \omega(\sigma)} S^{|\omega|}.$$

The only thing that remains is to interchange the two sums and perform the sum over σ. To interchange the sum, notice that a bond configuration ω lies in $\omega(\sigma)$ if and only if σ is such that $\sigma_x = \sigma_y$ for all vertices x and y which are connected in ω. Thus letting $\sigma(\omega)$ denote such an allowable spin configuration, we can write

$$Z_{[\sigma]} = \sum_\omega \sum_{\sigma(\omega)} S^{|\omega|}.$$

To perform the inner sum we simply need to count (given ω) how many $\sigma(\omega)$'s there are. Since each connected component must all have the same spin (and there are two choices in this case), there are exactly $2^{c(\omega)}$ such $\sigma(\omega)$. Thus

$$Z_{[\sigma]} = \sum_\omega 2^{c(\omega)} S^{|\omega|}.$$

Now multiply the above by $\left(\frac{1}{1+S}\right)^E$, where E is the total number of edges in Λ. Then we obtain

$$\begin{aligned} \left(\frac{1}{1+S}\right)^E Z_{[\sigma]} &= \sum_\omega 2^{c(\omega)} \left(\frac{S}{1+S}\right)^{|\omega|} \left(\frac{1}{1+S}\right)^{E-|\omega|} \\ &\equiv \sum_\omega 2^{c(\omega)} p^{|\omega|} (1-p)^{E-|\omega|} \\ &\equiv Z_{[\omega]}. \end{aligned}$$

Thus the Ising model at parameter J and the 2–state Potts model at parameter

$$p = \frac{e^J - 1}{e^J}$$

have (up to a constant) the same partition function. More precisely, we have a (stochastic) many–to–one map from a bond configuration ω to $2^{c(\omega)}$ ($q^{c(\omega)}$ in the q–state Potts case) spin configurations such that every site in a connected component of the bond configuration has the same spin. Or conversely, a spin configuration σ maps to $2^{I(\sigma)}$ bond configurations which has "occupied" bonds between neighboring sites *only if* they have the same spin. A good way to understand this interplay between spin and bond configurations is to couple together the two models to obtain a model of *both* spin and bond configurations:

$$W(\{\sigma\}, \{\omega\}) \propto \prod_{\langle x,y \rangle} (1-p)\delta_{\omega_{xy},0} + p\delta_{\omega_{xy},1}\delta_{\sigma_x,\sigma_y},$$

where $p \in [0,1]$ is some suitable parameter, $\omega_{xy} = 1, 0$ depending on whether the bond connecting x and y is occupied or vacant, respectively. This is the Edwards–Sokal "measure". See the first page of [1].

Questions about the spin system can now be translated into questions about the random cluster model, e.g. the probability that two sites x and y have the same spin is now the same as the probability that x and y are in the same connected component plus $1/2$ ($1/q$ in the q–state Potts case) times the probability that they are not in the same connected component.

References

[1] Robert G. Edwards and Alan D. Sokal. *Generalization of the Fortuin–Kastelyn–Swendsen–Wang Representation and Monte Carlo Algorithm*. Physical Review D, vol. 38, **6**, 2009-2012 (1988).