RC Expansion/Edwards–Sokal Measure

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Let $\Lambda \subset \mathbb{Z}^d$ be finite and consider the q–state Potts model with couplings $\{J_{x,y}\}$ and free boundary condition (no boundary condition). The Hamiltonian ($q = 2$ is Ising) is

$$\mathcal{H}(\sigma) = -\sum_{\langle x,y \rangle} J_{x,y}(\delta_{\sigma_x,\sigma_y} - 1),$$

where $\sigma_x \in \{1, \dots, q\}$. The partition function is

$$Z_{[\sigma]} = \sum_{\sigma} e^{-\beta \mathcal{H}(\sigma)} = \sum_{\sigma} \left(\prod_{\langle x,y \rangle} e^{\beta [J_{x,y}(\delta_{\sigma_x,\sigma_y} - 1)]} \right) \tag{1}$$

Now consider the random cluster model with parameter q and bond "weights" $\{p_{x,y}\}$ with partition function given by

$$Z_{[\omega]} = \sum_{\omega} q^{c(\omega)} \left(\prod_{\langle x,y \rangle : \omega_{x,y} = 1} (p_{x,y}) \prod_{\langle x,y \rangle : \omega_{x,y} = 0} (1 - p_{x,y}) \right), \tag{2}$$

where $c(\omega)$ denotes the number of connected components of the bond configuration ω. Note that

$$\begin{aligned} e^{\beta J_{x,y}(\delta_{\sigma_x,\sigma_y} - 1)} &= \delta_{\sigma_x,\sigma_y} + (1 - \delta_{\sigma_x,\sigma_y}) e^{-\beta J_{x,y}} \\ &= \delta_{\sigma_x,\sigma_y} p_{x,y} + (1 - p_{x,y}), \end{aligned}$$

if we set

$$p_{x,y} = 1 - e^{-\beta J_{x,y}}.$$

Plugging this into (1), we see that

$$e^{-\beta \mathcal{H}(\sigma)} = \prod_{\langle x,y \rangle} (\delta_{\sigma_x,\sigma_y} p_{x,y} + (1 - p_{x,y})). \tag{3}$$

If we expand out the product, we get

$$\sum_{\Omega\subset\Omega(\Lambda)}\left(\prod_{\langle x,y\rangle\in\Omega}(\delta_{\sigma_x,\sigma_y}p_{x,y})\prod_{\langle x,y\rangle\notin\Omega}(1-p_{x,y})\right),$$

where $\Omega(\Lambda)$ denotes the set of bonds connecting points in Λ. Each term above can be identified as a bond configuration where each bond is occupied if and only if it lies in Ω. The above sum can now be rewritten as

$$\sum_{\omega}\left(\prod_{\langle x,y\rangle:\omega_{x,y}=1}(\delta_{\sigma_x,\sigma_y}p_{x,y})\prod_{\langle x,y\rangle:\omega_{x,y}=0}(1-p_{x,y})\right).$$

The partition function (1) now becomes

$$\sum_{\sigma}\sum_{\omega}\left(\prod_{\langle x,y\rangle:\omega_{x,y}=1}(\delta_{\sigma_x,\sigma_y}p_{x,y})\prod_{\langle x,y\rangle:\omega_{x,y}=0}(1-p_{x,y})\right). \tag{4}$$

Observe that the above sum is finite and so far is simply summed over all possible bond and spin configurations, and hence we may interchange the order of summation to obtain

$$\sum_{\omega}\sum_{\sigma}\left(\prod_{\langle x,y\rangle:\omega_{x,y}=1}(\delta_{\sigma_x,\sigma_y}p_{x,y})\prod_{\langle x,y\rangle:\omega_{x,y}=0}(1-p_{x,y})\right). \tag{5}$$

From (4) we see that if ω is such that a bond is present when $\sigma_x\neq\sigma_y$, then the contribution to the sum is zero. Similarly, in (5), we see that if σ is such that $\sigma_x\neq\sigma_y$ when there is a bond present between x and y, then the contribution to the sum is zero. So to get rid of the $\delta_{\sigma_x,\sigma_y}$, we introduce the constraint function

$$\Delta(\sigma,\omega)=\Delta_\sigma(\omega)=\Delta_\omega(\sigma),$$

which is the indicator function of the fact that ω and σ are compatible in the sense that $\sigma_x=\sigma_y$ whenever $\omega_{x,y}=1$ and $\omega_{x,y}=0$ whenever $\sigma_x\neq\sigma_y$. We can now rewrite the Potts partition function as

$$Z_{[\sigma]}=\sum_{\omega}\sum_{\sigma}\Delta_\omega(\sigma)\left(\prod_{\langle x,y\rangle:\omega_{x,y}=1}(p_{x,y})\prod_{\langle x,y\rangle:\omega_{x,y}=0}(1-p_{x,y})\right).$$

Now we observe that given ω,

$$\sum_{\sigma}\Delta_\omega(\sigma)=q^{c(\omega)}, \tag{6}$$

since the Δ functions forces each connected cluster of ω to have the same spin and there are q possible spins. We can now perform the inner sum in the penultimate display to see that in fact (with $p_{x,y} = 1 - e^{-\beta J_{x,y}}$)

$$Z_{[\sigma]} = Z_{[\omega]} \equiv Z_{[\sigma,\omega]}.$$

So this quantity is the normalization constant for three (probability) measures: Potts, random cluster, and Edwards–Sokal, and from now on will simply be denoted Z. Explicitly, the Potts measure assigns a spin configuration σ the probability

$$\mu_P(\sigma) = \frac{1}{Z} \prod_{\langle x,y \rangle} (\delta_{\sigma_x,\sigma_y} p_{x,y} + (1 - p_{x,y})),$$

the random cluster measure assigns a bond configurations ω the probability

$$\mu_{RC}(\omega) = \frac{1}{Z} \times \prod_{\langle x,y \rangle} (\delta_{\omega_{x,y},1} p_{x,y} + \delta_{\omega_{x,y},0}(1 - p_{x,y}))$$

and the Edwards–Sokal measure assigns a spin–bond configuration (σ, ω) the probability

$$\mu_{ES}(\sigma,\omega) = \frac{1}{Z} \times \Delta(\sigma,\omega) \times \prod_{\langle x,y \rangle} (\delta_{\omega_{x,y},1} p_{x,y} + \delta_{\omega_{x,y},0}(1 - p_{x,y})). \quad (7)$$

By (6), if we fix ω and sum (7) over all spins σ, then we get exactly $\mu_{RC}(\omega)$ (so the marginal distribution of the bond variables is the RC model). Similarly, if we fix σ and sum (7) over all bonds (putting in $\delta_{\sigma_x,\sigma_y}$ and erasing $\Delta(\sigma,\omega)$), then we get $\frac{1}{Z}\prod_{\langle x,y \rangle}(1 - p_{x,y}) + p_{x,y}\delta_{\sigma_x,\sigma_y})$, which by (3) is exactly $\mu_P(\sigma)$ (so the marginal distribution of the spin variables is the Potts model).

Now let's find the conditional distributions. Let $g(\sigma)$ be a (dummy) spin observable. We compute the expected value of $g(\sigma)$, $\langle g(\sigma) \rangle_f^\Lambda$, with free boundary conditions in Λ. We would like to write it as a nested expectation $\mathbb{E}_\omega[\mathbb{E}_\sigma(g(\sigma) \mid \omega)]$. Writing out $\langle g(\sigma) \rangle_f^\lambda$ and expanding as before using (3), we have

$$
\begin{aligned}
\langle g(\sigma)\rangle_f^\Lambda &= \frac{1}{Z}\sum_\sigma e^{-\beta\mathcal{H}(\sigma)} g(\sigma)\\
&= \frac{1}{Z}\sum_\sigma \left(\prod_{\langle x,y\rangle}[\delta_{\sigma_x,\sigma_y}p_{x,y} + (1-p_{x,y}]\right) g(\sigma)\\
&= \frac{1}{Z}\sum_\sigma\sum_\omega \Delta_\sigma(\omega)\left(\prod_{\langle x,y\rangle}[\delta_{\omega_{x,y,1}}p_{x,y} + \delta_{\omega_{x,y,0}}(1-p_{x,y})]\right) g(\sigma)\\
&= \frac{1}{Z}\sum_\omega \left(\prod_{\langle x,y\rangle}[\delta_{\omega_{x,y,1}}p_{x,y} + \delta_{\omega_{x,y,0}}(1-p_{x,y})]\right)\sum_\sigma \Delta(\sigma,\omega) g(\sigma)\\
&= \frac{1}{Z}\sum_\omega \frac{q^{c(\omega)}}{\sum_\sigma \Delta(\sigma,\omega)}\left(\prod_{\langle x,y\rangle}[\delta_{\omega_{x,y,1}}p_{x,y} + \delta_{\omega_{x,y,0}}(1-p_{x,y})]\right)\sum_\sigma \Delta(\sigma,\omega) g(\sigma)\\
&= \sum_\omega \mu_{RC}(\omega)\left(\sum_\sigma \frac{\Delta(\sigma,\omega)}{\sum_\sigma \Delta(\sigma,\omega)} g(\sigma)\right),
\end{aligned}
$$

where we have used (6) ($\sum_\sigma \Delta(\sigma,\omega) = q^{c(\omega)}$). So we see that

$$\mu_{ES}(\sigma \mid \omega) = \frac{1}{q^{c(\omega)}}\Delta(\sigma,\omega),$$

that is, given ω, the conditional measure concentrates on σ's which are compatible with ω, assigning each such σ probability $1/q^{c(\omega)}$. Equivalently, given ω, the clusters of ω are labeled $1,\ldots,q$ with uniform probability. Here is a quick application of this: consider $q=2$ (the Ising case). Then

$$\langle \sigma_x\sigma_y\rangle_f^\Lambda = \sum_\omega \mu_{RC}(\omega)\left(\sum_\sigma \frac{\Delta(\sigma,\omega)}{2^{c(\omega)}}(2\delta_{\sigma_x,\sigma_y}-1)\right).$$

If x and y and in the same connected component of ω, then $\sigma_x=\sigma_y$ and the contribution of the inner sum is 1. If x and y are in different connected components, then with probability 1/2 (so half the contributing configurations) they will be labeled the same spin, in which case the inner sum contributes -1/2. Therefore the net contribution from bond configurations with x and y in different clusters is 0, and we conclude

$$\langle \sigma_x\sigma_y\rangle_f^\Lambda = \mu_{RC}(x \leftrightarrow y).$$

This formula can be generalized to integer values of q, if we take the "tetrahedron" representation, i.e. we represent the spin variables as unit vectors pointing to vertices of a $(q-1)$–dimensional "tetrahedron", so that we have

$$\sigma_x \cdot \sigma_y = (q\delta_{\sigma_x,\sigma_y} - 1)/(q-1).$$

Now let $g(\omega)$ be a (dummy) bond observable. Then we may "reverse" expand using (3) and (6) and get

$$\begin{aligned}
\langle g(\omega) \rangle &= \frac{1}{Z} \sum_\omega q^{c(\omega)} \left(\prod_{\langle x,y \rangle} [\delta_{\omega_{x,y,1}} p_{x,y} + \delta_{\omega_{x,y,0}} (1 - p_{x,y})] \right) g(\omega) \\
&= \frac{1}{Z} \sum_\omega \sum_\sigma \Delta(\sigma, \omega) \left(\prod_{\langle x,y \rangle} [\delta_{\sigma_x,\sigma_y} \delta_{\omega_{x,y,1}} p_{x,y} + \delta_{\omega_{x,y,0}} (1 - p_{x,y})] \right) g(\omega) \\
&= \frac{1}{Z} \sum_\sigma \sum_\omega \Delta(\sigma, \omega) \left(\prod_{\langle x,y \rangle : \omega_{x,y}=1} \delta_{\sigma_x,\sigma_y} p_{x,y} \right) \left(\prod_{\langle x,y \rangle : \omega_{x,y}=0} (1 - p_{x,y}) \right) g(\omega) \\
&= \frac{1}{Z} \sum_\sigma \sum_\omega \frac{\prod_{x,y} [p_{x,y} \delta_{\sigma_x,\sigma_y} + (1 - p_{x,y})]}{\sum_\omega \Delta(\sigma, \omega) \prod_{x,y} [\delta_{\omega_{x,y,1}} p_{x,y} + \delta_{\omega_{x,y,0}} (1 - p_{x,y})]} \\
&\times \prod_{x,y} [\delta_{\omega_{x,y,1}} p_{x,y} + \delta_{\omega_{x,y,0}} (1 - p_{x,y})] \Delta(\sigma, \omega) g(\omega) \\
&= \sum_\sigma e^{-\beta \mathcal{H}(\sigma)} \left(\sum_\omega \frac{\Delta(\sigma, \omega) \prod_{x,y} [\delta_{\omega_{x,y,1}} p_{x,y} + \delta_{\omega_{x,y,0}} (1 - p_{x,y})]}{\sum_\omega \Delta(\sigma, \omega) \prod_{x,y} [\delta_{\omega_{x,y,1}} p_{x,y} + \delta_{\omega_{x,y,0}} (1 - p_{x,y})]} g(\omega) \right).
\end{aligned}$$

So we see that

$$\mu_{ES}(\omega \mid \sigma) = \frac{\Delta(\sigma, \omega) \prod_{x,y} [\delta_{\omega_{x,y,1}} p_{x,y} + \delta_{\omega_{x,y,0}} (1 - p_{x,y})]}{\sum_\omega \Delta(\sigma, \omega) \prod_{x,y} [\delta_{\omega_{x,y,1}} p_{x,y} + \delta_{\omega_{x,y,0}} (1 - p_{x,y})]}.$$

Again the conditional measure is concentrated on bond configurations which are compatible with σ: Given σ, bonds are placed between x and y with probability $p_{x,y}$ if they have the same spin.

References

[1] M. Aizenman, J. T. Chayes, L. Chayes, and C. M. Newman. *Discontinuity of the Magnetization in One-Dimensional* $1/|x-y|^2$ *Ising and Potts Models.* J. Stat. Phys. **77**, 351-359 (1994).