On Convergence to SLE_6 II:

Discrete Approximations and Extraction of Cardy's Formula for General Domains

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Abstract: We show how to extract Cardy's Formula for a general class of domains given the requisite interior analyticity statement. This is accomplished by a careful study of the interplay between discretization schemes and extraction of limiting boundary values. Of particular importance to the companion work [3], we establish these results for slit domains and for the critical percolation models introduced in [6].

Keywords: Universality, conformal invariance, percolation, Cardy's Formula.

1 Introduction

In this note we wish to establish the validity of Cardy's Formula for crossing probabilities of certain 2D critical percolation models in a general (finite) domain $\Omega \subset \mathbb{C}$ (i.e., an open simply connected subset of $\mathbb{C}$). In this introduction we will not be overly concerned with model specifics, as the key point of this work is to clarify certain notions concerning discretization and extraction of appropriate boundary values. While these issues have been addressed to various extents in e.g., [18], [19], [5], [4], [15], and may seem quite self–evident – at least for nice (i.e., Jordan) domains, a complete and unified treatment for general domains appears to be absent. Moreover, aside from æsthetic appeal, the generality that appears here is certainly needed for the approach of proving convergence to SLE_6 outlined in [17] (see also [19]) and carried out in [3]. Our efforts will culminate in the establishment of Theorem 5.8 and Corollary 5.11 (which is stated in [3] as Lemma 2.6).

Since it is our intention that this note be self–contained, let us first review the methodology – introduced in [18] and adapted to the models in [6] (see also [3], §4.1) – by which Cardy's Formula can be extracted. At the level of the continuum we are interested in a domain $\Omega \subset \mathbb{C}$ which is a conformal triangle with boundary components $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ and marked prime ends (boundary points) $\{a, b, c\}$ – all in counterclockwise order – which represent the intersection of neighboring components. At the level of the lattice, at spacing ε, we consider an approximate domain Ω_ε, in which the percolation process occurs and which tends – in some sense – to Ω as $\varepsilon \to 0$. At the ε–scale, the competing (dual) percolative forces will be denoted by "yellow" and "blue".

Let z be an interior point (e.g., a vertex) in Ω_ε. We define the discrete crossing probability function $u_\varepsilon^B(z)$ to be probability that there is a blue path connecting $\mathcal{A}$ and $\mathcal{B}$, separating z from $\mathcal{C}$, with similar definitions for $v_\varepsilon^B(z)$ and $w_\varepsilon^B(z)$ along with yellow versions of these functions. For these objects, standard arguments show that subsequential limits exist; two seminal ingredients are required: First, they converge to harmonic functions with a particular conjugacy relation between them in the interior and second they satisfy certain ("obvious") boundary values. With these ingredients in hand it can be shown that the limiting functions are the so called Carleson–Cardy functions. E.g.,

$$\lim_{\varepsilon \to 0} u_\varepsilon^Y = u,$$

and similarly for the v's and w's where, e.g., according to [2], the functions u, v, w are such that

$$F := u + e^{2\pi i/3}v + e^{-2\pi i/3}w$$

is the unique conformal map from Ω to the equilateral triangle formed by the vertices 1, $e^{\pm 2\pi i/3}$. This is equivalent to Cardy's Formula.

We carry out the above scheme in its entirety for a general class of domains *and* their discrete approximations which is suitable for our uses in [3], Lemma 2.6/Corollary 5.11.

Remark. The appropriate discrete conjugacy relations for the $u_\varepsilon, v_\varepsilon$ and w_ε have only been established for the 2D triangular site models in [18] and the extension introduced in [6]. However, since the RSW estimates are purportedly universal and actually hold for any reasonable critical 2D percolation model, in principle we always have limiting functions u, v, w with some boundary values. Hence most of the content of the present work should apply. However, certain provisos and clarifications will be required; see Remark 5.6.

In the ensuing arguments we will have occasion to make use of the uniformization map $\varphi : \mathbb{D} \to \Omega$ (where $\mathbb{D}$ denotes the unit disk) provided by the Riemann Mapping Theorem. Here we will take φ to be normalized so that $\varphi(0) = z_0 \in \Omega$ for some fixed point z_0 well in the interior of Ω and $\varphi'(0) > 0$. We will also identify points on $\partial\mathbb{D}$ with boundary prime ends of $\partial\Omega$, via the Prime End Theorem. We refer the reader to e.g., [14] for such issues. Finally, the reader may wish to keep in mind that the reason for addressing most of the issues herein is for application to the case where the curves/slits under consideration are percolation interfaces/explorer paths; for discussions on this topic we refer the reader to [3].

2 The Carathéodory Minimum

We start by reviewing a standard notion of domain convergence, namely, Carathéodory convergence, mainly to phrase it in terms of more elementary conditions which are more convenient for our purposes. The reader can find similar conditions/discussions in e.g., Section 1.4 of [14].

Our general situation concerns a sequence of domains (Ω_n) which converge in some sense to the limiting Ω along with functions (u_n, v_n, w_n) converging to a harmonic triple (u, v, w) satisfying the appropriate conjugacy relations. As a minimal starting point let us consider the following pointwise (geo)metric conditions for domain convergence:

(i_I) If $z \in \Omega$, then $z \in \Omega_n$ for all n sufficiently large.

(i_{II}) If $z_n \in \Omega_n^c$, then all subsequential limits of (z_n) must lie in Ω^c.

(e) For all $z \in \Omega^c$ (including, especially $\partial\Omega$) there exists some sequence $z_{n_k} \in \Omega_{n_k}^c$ such that $z_{n_k} \to z$.

Conditions (i_I) and (i_{II}) ensure that limiting values of u, v and w in (the interior of) Ω can be retrieved and are defined by values of u_n inside Ω_n whereas condition (e) implies that Ω_n's don't converge to a domain strictly larger than Ω, so that the boundary values of u on $\partial\Omega$ might actually correspond to (the limit of) boundary values of u_n on Ω_n. Indeed, these preliminary conditions turn out to be equivalent to Carathéodory convergence (see e.g., [9]; although in our context we will actually not have occasion to use convergence of the relevant uniformization maps). More precisely, first we have the following result, whose proof is elementary (and we include for completeness):

Proposition 2.1. *Consider domains $\Omega_n, \Omega \subset \mathbb{C}$ all containing some point z_0. Then the following are equivalent:*

1. *If K is compact and $K \subset \Omega$, then $K \subset \Omega_n$ for all but finitely many Ω_n.*
2. (i_I) *For all $z \in \Omega$, $z \in \Omega_n$ for all but finitely many Ω_n.*

   (i_{II}) *If $z_n \in \Omega_n^c$, then all subsequential limits of (z_n) must lie in Ω^c.*

3. If $z \in \Omega$, and $\delta < d(z, \partial\Omega)$, then $B_\delta(z) \subset \Omega_n$, for all but finitely many Ω_n.

Proof. 1 ⇒ 2) To see (i_I) suppose $z \in \Omega$ and $d(z, \partial\Omega) > \delta$, then $\overline{B_\delta(z)} \subset \Omega$ and is compact and hence we have $\overline{B_\delta(z)} \subset \Omega_n$ for all n sufficiently large and hence $z \in \Omega_n$ for all n sufficiently large; conversely, To see (i_{II}), suppose $z_n \to z$ with $z_n \in \Omega_n^c$ and suppose towards a contradiction that $z \in \Omega$. Then again arguing as before, $\overline{B_\delta(z)} \subset \Omega_n$ for n sufficiently large, but then $z_n \in B_\delta(z)$ also for n even larger, which implies that these $z_n \in \Omega_n$, a contradiction.

2 ⇒ 3) Again suppose $d(z, \partial\Omega) > \delta$ so that $\overline{B_\delta(z)} \subset \Omega$. If it is not the case that $B_\delta(z) \subset \Omega_n$ for n sufficiently large, then we can find a sequence $z_n \in B_\delta(z) \cap \Omega_n^c$. Since $\overline{B_\delta(z)}$ is compact, there exists a subsequential limit point $z_{n_k} \to z_*$, but then by (i_{II}), $z_* \notin \Omega$, contradicting $\overline{B_\delta(z)} \subset \Omega$.

3 ⇒ 1) Let $K \subset \Omega$ be compact. We can cover K by $K \subset \bigcup_{x \in K} B_{\delta_x}(x)$, with $\delta_x < d(x, \partial\Omega)$. By the assumed compactness, there is a finite subcover $K \subset \bigcup_{i=1}^{k} B_{\delta_{x_i}}(x_i)$. By 3), for $1 \leq i \leq k$, there exists N_i such that $B_{\delta_{x_i}}(x_i) \subset \Omega_n$ for all $n \geq N_i$, and hence it is the case that $K \subset \Omega_m$ for all $m > \max\{N_1, N_2, \ldots, N_k\}$. □

Now the notion of *kernel convergence* requires, in addition (specifically to condition 1 in the above Proposition) that Ω is the *largest* (simply connected) domain satisfying the above conditions. The addition of condition (e) indeed corresponds to maximality; arguments similar to those just presented easily lead to the following (whose proof is elementary and is also included for completeness):

Proposition 2.2. *The conditions (i_I), (i_{II}), (e) are equivalent to Ω_n converging to Ω in the sense of kernel convergence.*

Proof. In light of the above discussion, it is sufficient to show that the condition (e) is equivalent to the maximality condition on Ω required by kernel convergence.

⇒) Suppose Ω is not maximal and hence $\Omega \subsetneq \Omega'$ where Ω' satisfies (i_I) and (i_{II}). It must be the case then there is a point $z \in \partial\Omega \cap \Omega'$. By condition (e) there exists $z_{n_k} \to z$ with $z_{n_k} \in \Omega_{n_k}^c$, but condition (i_{II}) for Ω' implies that $z \in (\Omega')^c$, a contradiction.

⇐) Conversely, suppose Ω is maximal and assume towards a contradiction that Ω does not satisfy (e), so that there exists some point $z \in \Omega^c$ and some $\delta > 0$ such that $B_\delta(z) \subset \Omega_n$ for all n sufficiently large. By the maximality of Ω, it must be the case that $\overline{B_\eta(z)} \subset \Omega$ for any $\eta < \delta$, which implies in particular that $z \in \Omega$, a contradiction. □

It is noted that in the present setting of bounded, simply connected domains, kernel convergence is, by the theorem of Carathéodory, equivalent to convergence uniformly on compact sets of the corresponding uniformization maps (see e.g., [9], Theorem 3.1). The latter notion is known as *Carathéodory convergence* and we will use this terminology throughout.

As is perhaps already clear, Carathéodory convergence alone is insufficient for our purposes: Since the functions u, v, w must acquire prescribed boundary values on separate pieces of $\partial\Omega$, it is manifest that (some notion of) separate convergence of the corresponding pieces of the boundary in $\partial\Omega_n$ will be required. Special attention is needed for the cases of domains with slits – which are of seminal importance when we consider the problem of convergence to SLE_κ. The situation is in fact rather subtle: Note that in both Figure 1 and Figure 3, we have that Ω_ε Carathéodory converges to Ω, but whereas the situation in Figure 1 disrupts establishment of the proper boundary value, the situation in Figure 3 is perfectly acceptable (see Remarks 3.2 and 4.3).

3 Interior Approximations

We will begin by considering the *interior* approximations, where $\Omega_\varepsilon \subset \Omega$ for all ε. For earlier considerations along these lines, see [8] and [10]. Here, the crucial advantage is that all the Ω_ε's can be viewed under a *single* uniformization map; this allows for relatively simple resolutions of various concerns of a geometric/topological nature. Moreover, this appears to be the simplest setting for the purposes of establishing Cardy's Formula in a fixed (static) domain, i.e., where Ω is fixed for once and for all and we are free to generated Ω_ε in any fashion. (See especially Example 3.3 below.) In particular, for circumstances where the fixed domain problem is all that is of interest, the reader is invited to skip Section 4 altogether. We start with

Definition 3.1 (Interior Approximations)**.** We call $(\Omega_\varepsilon^\bullet)$ an *interior approximation* to Ω if:

(I) The domains $\Omega_\varepsilon^\bullet$ consist of one or more (graph) connected components (the latter case is an artificial effect of the lattice spacing being not fine enough). Each component is bounded by a closed polygonal path, and the union of all such polygonal paths we identify as the boundary $\partial\Omega_\varepsilon^\bullet$. In particular, $\partial\Omega_\varepsilon^\bullet$ consists exclusively of polygonal edges each of which is a portion of the border of an element in $(\Omega_\varepsilon^\bullet)^c$.

(II) The boundary $\partial\Omega_\varepsilon^\bullet$ is divided disjoint segments, denoted by $\mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon, \dots$ in (rough) correspondence with the (finitely many) boundary components $\mathcal{A}, \mathcal{B}, \dots$ of the actual domain Ω. In case Ω_ε is a single component, these are joined at vertices $a_\varepsilon, b_\varepsilon, \dots$ corresponding to the appropriate marked prime ends. In the multi–component case, if necessary, a similar procedure may be implemented, implying the possible existence of several a_ε's etc. When required, *the* $a_\varepsilon, b_\varepsilon, \dots$, etc., will be the one corresponding to the "principal" component of Ω_ε, namely, the component which contains the point z_0, which, we recall, served to normalize the uniformization map. Here it is tacitly assumed that ε is small enough so that this component has a representative of each type.

Further, we require the following:

(i) It is always the case that $\Omega_\varepsilon^\bullet \cup \partial\Omega_\varepsilon^\bullet \subset \Omega$. That is, $\Omega_\varepsilon^\bullet$ is in fact a *strictly* inner approximation.

This property ensures that indeed all of the $\overline{\Omega}_\varepsilon$ can be viewed under the (single) conformal map $\varphi : \mathbb{D} \to \Omega$ in the ensuing arguments.

(ii) Each $z \in \Omega$ lies in $\Omega_\varepsilon^\bullet$ for all ε sufficiently small.

It can be seen that conditions (i) and (ii) imply that for any $z \in \partial\mathcal{A}$, there exists some sequence $z_\varepsilon \to z$ with $z_\varepsilon \in \mathcal{A}_\varepsilon$, and similarly for $\mathcal{B}$, etc.

(iii) Given any sequence (z_ε) with $z_\varepsilon \in \mathcal{A}_\varepsilon$ for all ε, any subsequential limit must lie in $\mathcal{A}$. Moreover, this must be true in the stronger sense that for any subsequential limit $\varphi^{-1}(z_{\varepsilon_n}) \to \zeta \in \partial\mathbb{D}$ then $\zeta \in \varphi^{-1}(\mathcal{A})$. Similarly for $\mathcal{B}$, etc.

In particular, any subsequential limit of the (a_ε)'s will converge to a point in a, and similarly for b, etc.

Remark 3.2.

- To avoid confusion, by the above method, an interior approximation to any slit domain – no matter how smooth the slit – necessarily consist of at least a small cavity of a few lattice spacings. It is noted that the explorer process itself produces just such a cavity.
- It is easy to check that interior approximations satisfy conditions (i_I), (i_{II}), (e).
- Condition (iii) is indeed used to ensure that the limiting boundary values are unambiguous and correspond to the desired result (see Lemma 5.2). A simple scenario where careless approximation leads to the wrong boundary value is illustrated in Figure 1.
- Note that even though for convenience we have assumed in (iii) that $z_\varepsilon \in \mathcal{C}_\varepsilon$ and have used the uniformization map φ, what is sufficient is that if $z_k \to z \in \mathcal{C}$, then for all but finitely many k, z_k should be close to $\mathcal{C}_\varepsilon$, in some appropriate sense. Indeed, we shall have occasion to formulate such a definition later, for the statement of Lemma 4.4.

Example 3.3. An example of an interior approximation is what we will call the *canonical approximation*, constructed as follows. To be definitive, consider a tiling problem with finitely many types of tiles. We formally define the scale ε to be the maximum diameter of any tile. As usual, we may regard all of $\mathbb{C}$ as having been tiled – "$\mathbb{C}_\varepsilon$". The domain Ω_ε is defined as precisely those tiles in $\mathbb{C}_\varepsilon$ which are entirely (including their boundary) in Ω. Clearly then this construction satisfies (i); condition (ii) is also satisfied: if $z \in \Omega$ is such that $d(z, \partial\Omega) > \varepsilon_0$, then $z \in \Omega_\varepsilon$ for all $\varepsilon < \varepsilon_0$.

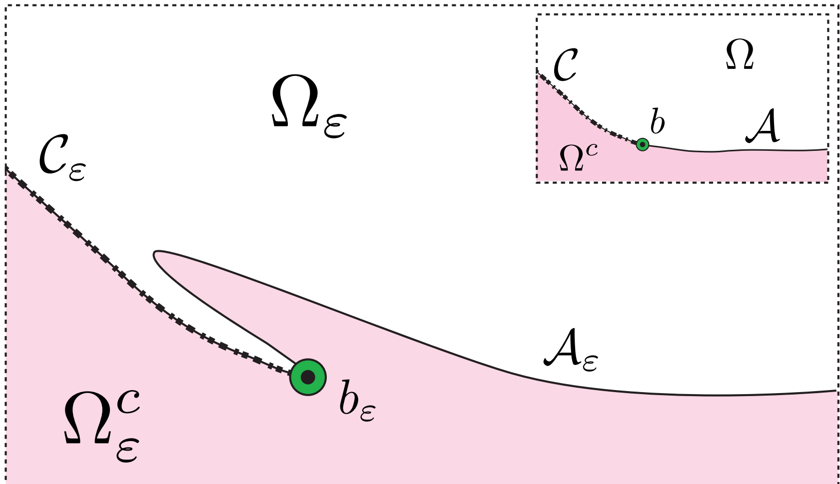

Figure 1: Violation of condition ii) in Definition 3.1, which would lead to incorrect (limiting) boundary values.

At this stage $\partial\Omega_\varepsilon$ is just one or more closed polygonal paths. The boundary component types are determined as follows: For the marked points, e.g., a, consider the neighborhood $Q_\delta(a)$ defined as follows: Let $\mathfrak{c}_\varepsilon$ denote a sequence of crosscuts of $\varphi^{-1}(a)$ with the property that $\varphi(\mathfrak{c}_\varepsilon)$ contains a δ neighborhood of a with $\delta/\varepsilon \to \infty$ and $\delta(\varepsilon) \to 0$; $Q_\delta(a)$ is then the set bounded by $\varphi(\mathfrak{c}_\varepsilon)$ and the relevant portion of $\partial\Omega$. It is clear, for ε small, that "outside" these neighborhoods, the assignment of boundary component type is unambiguous. Here we say a boundary segment is "outside" $Q_\delta(a)$ etc., if all tiles (intersecting Ω) touching the segment in question lie in the complement of $Q_\delta(a)$. Indeed, each segment of $\partial\Omega_\varepsilon$ belongs to a tile that intersects the boundary. For a fixed element of $\partial\Omega_\varepsilon$ satisfying the above definition of "outside", *some* of the external tile is in Ω and therefore under φ^{-1}, the image of this portion of the tile joins up with $\partial\mathbb{D}$; furthermore, it joins with a unique boundary component image due to the size of the obstruction provided by $Q_\delta(a)$. Finally, inside these neighborhoods $Q_\delta(a)$, etc., all that must be specified are the points a_ε, etc., which as discussed above, may have multiple designations (due to the possibility of multiple components for Ω_ε). The rest of the boundary is then assigned accordingly.

Finally, let us establish (iii):

Claim 3.4. *The canonical approximation satisfies (iii).*

Proof. Let $z_\varepsilon \in \mathcal{A}_\varepsilon$ with some subsequential limit z. It is clear that $z \notin \Omega$ since all $z \in \Omega$ are a finite distance from the boundary while $d(z_\varepsilon, \partial\Omega) \leq \varepsilon$ by construction. Moreover, $z \in \mathcal{A}$ since $d(z_\varepsilon, \mathcal{A})$ is (generally less than ε but certainly) no larger than $\delta(\varepsilon)$. It remains to show the stronger statement that any subsequential limit of $\varphi^{-1}(z_\varepsilon)$ is in $\varphi^{-1}(\mathcal{A})$. If $\zeta_\varepsilon = \varphi^{-1}(z_\varepsilon)$ converges to the image of a *marked* (end)point in $\varphi^{-1}(\mathcal{A})$ there is nothing to prove. Thus we may assume that for any κ, eventually ζ_ε is outside the κ–neighborhood of the marked (end)points, which we temporarily denote by $\alpha_1, \alpha_2 \in \varphi^{-1}(\mathcal{A})$. Now let $\eta < \kappa$ be such that the η neighborhood of $\partial\mathbb{D} \setminus [B_\kappa(\alpha_1) \cup B_\kappa(\alpha_2)]$ consist of two disjoint components, one containing all of the rest of $\varphi^{-1}(\mathcal{A})$ and the other associated with $\varphi^{-1}(\partial\Omega \setminus \mathcal{A})$. Finally consider the neighborhood

$$M_\eta := \mathcal{N}_\eta(\mathcal{A}) \cap \varphi[\mathcal{N}_\eta(\varphi^{-1}(\mathcal{A})],$$

where $\mathcal{N}_\eta(\cdot)$ denotes the Euclidean η neighborhood of $(\cdot)$ in the appropriate relative topology. Since it is agreed that z_ε stays outside $\varphi(B_\kappa(\alpha_1) \cup B_\kappa(\alpha_2))$ it is clear that, for all ε sufficiently small, $z_\varepsilon \in M_\eta$ and therefore $\zeta_\varepsilon \in \mathcal{N}_\eta(\varphi^{-1}(\mathcal{A})) \setminus [B_\kappa(\alpha_1) \cup B_\kappa(\alpha_2)]$ and not in the complementary η band described above. It follows that the limit must be in $\varphi^{-1}(\mathcal{A})$.

□

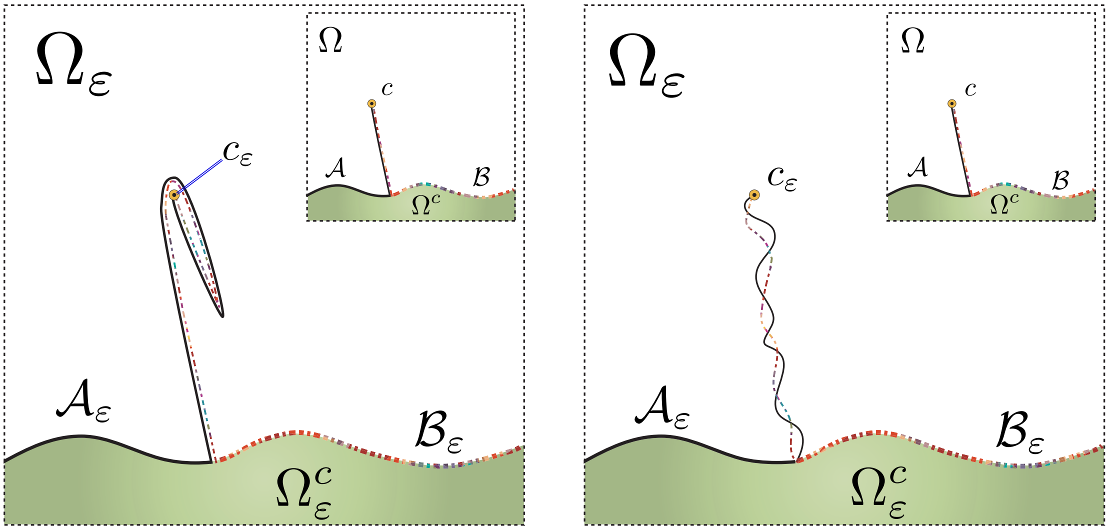

Figure 2: Masking and intermixing of boundary values.

4 Sup–Approximations

Unfortunately, for various purposes, e.g., certain proofs of convergence to SLE_6, we will need slightly more generality than the internal approximations as provided in Definition 3.1. Specifically, a situation may arise where we have domains Ω_n (given, e.g., stochastically) which are converging, in some sense, to Ω. One must then contemplate an ε–approximate for these Ω_n's – and hence Ω – and extract some (diagonal) subsequence. However it soon becomes clear that interior approximations will in general be insufficient: Explicitly, it may be the case that $(\Omega_n)_{\varepsilon_m}$ is *not* an interior approximation to Ω, no matter how small ε_m is. These circumstances can and will readily occur in the pertinent case of slit domains.

Eventually, we will resolve this problem and indeed obtain such a "diagonal" discretization statement (see Proposition 5.10) by studying more general discretization schemes which are commensurate with the nature in which Ω_n converges to Ω. Here, informally, we will describe the two additional properties which are essential in our more general context:

• Actual sup–norm convergence of separate sides of the slits (which in the discrete approximations may well be separate curves): This is to prevent the masking of one boundary value by another near the joining of boundaries.

• The *well–organization* property: This is to prevent confusion of boundary values that could be caused by intermingling (crisscrossing) of the two curves approximating the opposite sides of the slit.

Scenarios in violation of these properties are depicted in Figure 2.

Remark 4.1. If γ_1 and γ_2 are two curves, then as usual the sup distance between them is given as

$$\mathrm{dist}(\gamma_1, \gamma_2) = \inf_{\varphi_1,\varphi_2} \sup_t |\gamma_1(\varphi_1(t)) - \gamma_2(\varphi_2(t))|.$$

For certain purposes, it is pertinent to consider weighting the sup–norms of portions of the curves in accord with the particular crosscut in which the portion resides. We will denote the associated distance by **Dist**; see [3], §3.2 for the definition and discussions. However, our ensuing arguments will not be sensitive as to whether we are using the original sup–norm or the weighted version and thus we will continue to use the sup–norm.

Definition 4.2 (Sup–approximations)**.** Suppose $\partial\Omega$ can be further divided (perhaps by other marked boundary points) with the boundary between *these* points described by Jordan arcs or, more generally, Löwner curves. We shall label the new points J_1, J_2, etc., and between certain pairs, e.g., J_k & J_{k+1} will be a Löwner curve denoted by $[J_k, J_{k+1}]$. The marked prime ends $a, b, \ldots$ may serve as an endpoint of (some of) these segments, but it is understood that they do not reside *inside* these arcs.

Some of this curve (often enough all of it) will be part of the boundary $\partial\Omega$. (On the other hand, it can be envisioned that a portion of this curve lives in a "swallowed" region and is part of Ω^c.) At the discrete level, we recall that $\partial\Omega_\varepsilon$ is automatically a union of closed self–avoiding curves. It will be supposed that Ω_ε has corresponding $J_1^\varepsilon, J_2^\varepsilon, \dots$ and the relevant portion of the curve between the relevant J–pair converges in sup–norm to the corresponding portions in $\partial\Omega$ – or Ω^c, as the case may be – at rate $\eta(\varepsilon)$.

We assume that all of this transpires in such a way that the following property, which we call *well organized*, holds: For any curve of interest $[J_k^\varepsilon, J_{k+1}^\varepsilon]$, pick points p and p' on this arc. Consider δ–neighborhoods around p and p' and consider the portion of the arc joining these neighborhoods (last exit from neighborhood around p to first entrance to neighborhood around p'), which we label $\mathscr{L}$. Let $\mathscr{P}$ be any path connecting the boundaries of these neighborhoods to one another in the complement of all $\partial\Omega_\varepsilon$. Then the relevant portions of $\partial B_\delta(p)$, $\partial B_\delta(p')$, $\mathscr{P}$ and $\mathscr{L}$ clearly form a Jordan domain, whose interior we denote by $\mathcal{O}$. Let $\mathcal{O}' \subset \mathcal{O}$ denote the connected component of $\mathscr{P}$ in $\overline{\mathcal{O}} \setminus \partial\Omega_\varepsilon$. Then, $\partial\mathcal{O}' \cap \partial\Omega_\varepsilon$ is *monochrome*, i.e., it cannot intersect both $[J_k^\varepsilon, J_{k+1}^\varepsilon]$ and $[J_\ell^\varepsilon, J_{\ell+1}^\varepsilon]$ for $k \neq \ell$. While this may sound overly complicated, what we have in mind is actually a simple topological criterion, c.f., Remark 4.3.

The rest of the domain and boundary is approximated by interior approximation. Thus, for those J_k's which divide arc–portions of $\partial\Omega$ from "other", we require *commensurability* at the joining points. In particular, in order that the interior approximation be implementable, it is clear that we must require $J_k^\varepsilon \in \Omega$.

Remark 4.3.

• While at first glance it is difficult to imagine that $\partial\mathcal{O}'$ is anything except, say $\mathscr{L}$, what we have in mind is when $\mathscr{L}$ and a neighboring curve are some approximation to a two–sided slit. A crisscrossing approximation can very well lead to incorrect limiting boundary values – or none at all. The well–organized property does not permit the sides of the approximation to crisscross one another. Alternatively, this is a simple topological criterion which can be rephrased as follows: Under the uniformization map (in fact any homeomorphism onto a Jordan domain would do) the image of each of these J–pieces occupies a single contiguous piece of the boundary. This sort of monochromicity property is required for well–behaved convergence of relevant boundary conditions which we shall need later. It is clear that this well–organized property is satisfied by the trace of any discrete percolation explorer process.

• Sup–approximations satisfy conditions (i_I), (i_{II}), (e).

• The added difficulty here is that since the approximation is no longer interior, we can no longer determine the "topological situation" by looking under a single conformal map. E.g., for a point close to the boundary, we can no longer determine which boundary piece it is "really" close to. This is exemplified by the case of a slit domain: If, say, part of $\mathcal{C}$ is one side of a two–sided slit γ, then points close to γ on one side (corresponding to $\mathcal{C}$) will have small u value which tends to 0 whereas points close to γ on the other side (corresponding to say $\mathcal{B}$) will tend to non–trivial boundary values. In the case of interior approximation all such ambiguities were resolved by looking under the conformal map φ^{-1}.

• It is worth noting that the important case in point where the boundary consist of an original Ω with a (Löwner) slit – which might be two sided – falls into the setting under consideration. In particular, we will have occasion to consider cases where we have $\gamma_n \to \gamma$ in the sup–norm with γ_n being discrete explorer paths. In this case to check condition (i_{II}), we observe that if $\gamma_n \to \gamma$ in sup–norm and $z_n \in \gamma_n$ and $z_n \to z$, then $z \in \gamma$ and hence certainly in the complement of the the domain of interest $\Omega \setminus \mathbb{I}(\gamma)$ (i.e., Ω delete γ together with components "swallowed" by γ). For an illustration see Figure 3. These circumstances may be readily approximated by a hybrid of sup– and canonical approximations and, as is not hard to see, satisfy the condition of commensurability.

The main addition is the following lemma which serves the role of condition (iii) in Definition 3.1 to ensure unambiguous retrieval of boundary values (see Lemma 5.3). That is, if $w \in \Omega$ is close to $\mathcal{C}$ in the "homotopical sense" that any *short* walk from w which hits $\partial\Omega$ must hit the $\mathcal{C}$ portion of $\partial\Omega$ then w is close to $\mathcal{C}_n$ by the same criterion. A precise statement of this intuitive notion is, unfortunately, much more involved.

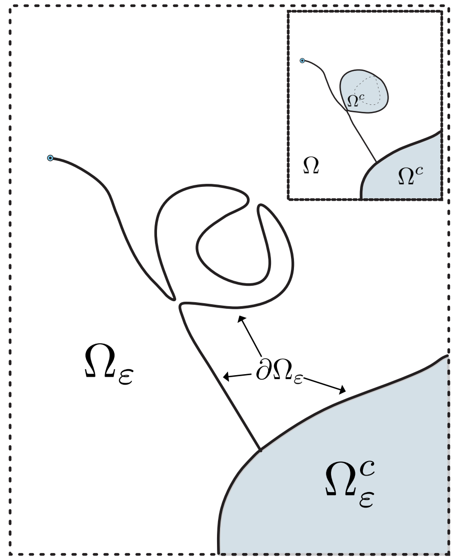

Figure 3: A case where the limiting domains does not contain a component present in approximating domains. Due to frequent self–touching, such (limiting) domains are in fact typical of SLE_6.

Lemma 4.4 (Homotopical Consistency)**.** *Consider a domain Ω with marked boundary prime ends $a, b \in \partial\Omega$. Let us focus on boundary $\mathcal{C}$ with end points a and b which we consider to be the bottom of the boundary. (Note that $\mathcal{C}$ may consist of Jordan arcs together with arbitrary parts – if double–sided slits are involved, such that not both sides belong to $\mathcal{C}$, then the corresponding arc(s) must be connected all the way up to b and/or a). Let us denote the sup–approximation to Ω by Ω_n and the portion of the boundary approximating $\mathcal{C}$ by $\mathcal{C}_n$.*

Suppose we have a point w which is more than Δ away from a and b and $\delta^\star$ away from $\mathcal{C}$ with $\Delta \gg \delta^\star$, such that $\vartheta = \varphi^{-1}(w)$ is close to $\varphi^{-1}(\mathcal{C})$. Then there exists $\eta > 0$ with $\eta \ll \delta^\star$ such that if $\operatorname{dist}(\mathcal{C}_n, \mathcal{C}) < \eta$ (here dist *denotes e.g., the sup–norm distance where appropriate, and otherwise the Hausdorff distance) then there exists some path $\mathscr{P}$ from (some point in) $\varphi^{-1}(B_\Delta(a))$ to (some point in) $\varphi^{-1}(B_\Delta(b))$ (we denote this by $\varphi^{-1}(B_\Delta(a)) \rightsquigarrow \varphi^{-1}(B_\Delta(b))$) such that in the sup–approximation Ω_n, w is in the bottom component of $\Omega_n \setminus \varphi(\mathscr{P} \cup B_\Delta(a) \cup B_\Delta(b))$ and further, any walk from w in the bottom component which hits $\partial\Omega_n$ must hit $\mathcal{C}_n$.*

Proof. For clarity, we divide the proof into four parts.

1. We let $\eta \ll \Delta \ll 1$ and consider, under the uniformization map, the set

$$\mathscr{B} := \mathbb{D} \setminus [\varphi^{-1}(B_\Delta(a)) \cup \varphi^{-1}(B_\Delta(b))].$$

Let us now draw a path $\mathscr{P}' : \partial\varphi^{-1}(B_\Delta(a)) \rightsquigarrow \partial\varphi^{-1}(B_\Delta(b))$ which defines top and bottom components in $\mathscr{B}$ with ϑ in the bottom component, and hence also the bottom component of $\varphi(\mathscr{B}) \setminus \varphi(\mathscr{P}'))$. Further, $\mathscr{P} := \varphi(\mathscr{P}')$ is some finite distance $\delta \gg \eta > 0$ away from w. (In essence, δ will now play the role of $\delta^\star$ in the statement of the Lemma.)

2. We now look at the domain

$$\mathscr{V}_n = \Omega_n \setminus [B_\Delta(a) \cup B_\Delta(b) \cup \mathscr{P}].$$

We claim that for n sufficiently large, all of the above is well–defined: Indeed, $\mathscr{P}$ is a compact set in Ω and hence for n sufficiently large, is contained in Ω_n, by Proposition 2.1. Of course, Ω_n itself may have many components; we are focusing on the principal component. Even so, with the above setup, $\mathscr{V}_n$ may also have many components, e.g., near the boundaries of $B_\Delta(a)$ and $B_\Delta(b)$ (see Figure 4).

However, we claim that it has the analogue of a top and bottom component: Indeed, it is clear that "large" compact sets in $\mathscr{B}$ well away form the boundaries continue to lie in large connected components of $\mathscr{V}_n$. More quantitatively, while at the scale Δ, $\partial\Omega_n$ may create various components by entering and re–entering $B_\Delta(a)$ and $B_\Delta(b)$, since $\mathcal{C}_n$ and $\mathcal{C}$ are η–close (say in the Hausdorff distance), if we shrink these

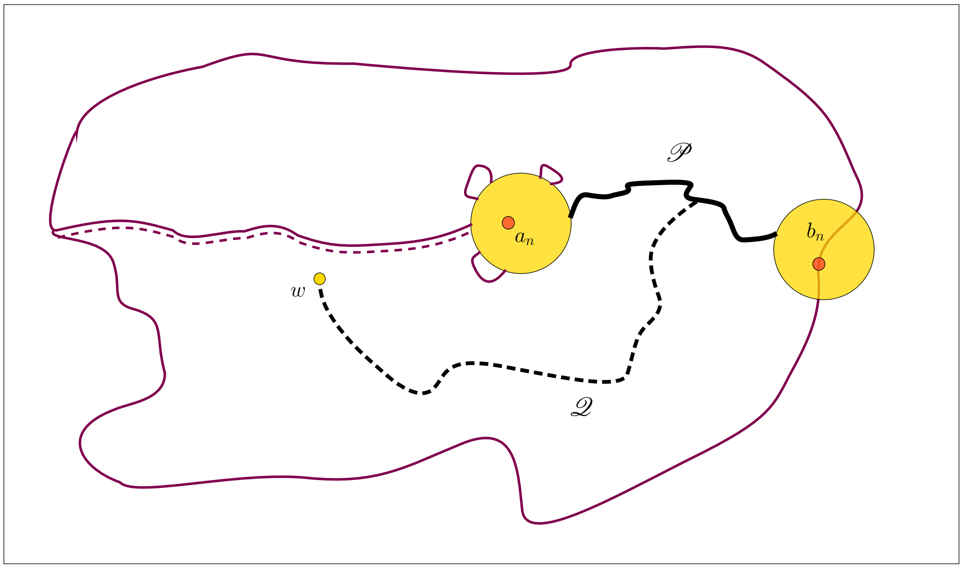

Figure 4: The domain $\mathscr{V}_n$, etc.

neighborhood balls to scale $\Delta - 2\eta$, then such components merge into (the) two principal components, leaving only η–scale small components in the vicinity of the neighborhood balls.

Finally, we claim that w is in the bottom component of $\mathscr{V}_n$. First, since $B_\delta(w)$ must all be in the same component of $\mathscr{V}_n$, w cannot be in a small η–scale component. The argument can be finished by any number of means. For example we may choose to regard $\mathscr{P}$ as two–sided; the component of w is determined by which side of $\mathscr{P}$ it may be connected to. For future reference, let $\mathscr{Q}' \subset \mathbb{D}$ denote a simple path (staying well away from $\partial\mathscr{B}$) connecting ϑ to $\mathscr{P}'$ and $\mathscr{Q}$ the image of $\mathscr{Q}'$ under φ. Then, again, by Proposition 2.1, for all n sufficiently large, the entirety of $\mathscr{Q}$ is found in Ω_n and the appropriate component – bottom – for w is determined for once and all. The relevant domains, etc., are illustrated in Figure 4.

3. It is clear that w is close to $\mathcal{C}_n$. We further claim that it is not obstructed from $\mathcal{C}_n$ by other portions of $\partial\Omega_n$, as may be the worry when a portion of $\mathcal{C}$ is (one side of) a two–sided slit. We need to divide into a few cases. First if the only portion of $\mathcal{C}$ which is close to q is approximated interiorly, then by an investigation of the situation under the uniformization map, it is clear that no obstruction is possible. So now we suppose that w is close to some Jordan arc $\mathscr{J} := [J_k, J_{k+1}]$. If $\mathscr{J}$ is one–sided, then there is no problem, since then w is not close in anyway to any other portion of the boundary except near the endpoints, which we may assume, by shrinking relevant scales if necessary, that w is far away from.

4. We are down to the main issue where $\mathscr{J}$ is a two–sided slit, which is being sup–norm approximated by $\mathscr{J}_n$. Since at least one of the end points must be a or b, let us assume without loss of generality that $J_k = a$. We will need to do some refurbishing, starting with the neighborhood balls around a (and b, if necessary). Let $\mathfrak{q}$ be a point on $\mathscr{J}$ near a. It is manifestly the case that $\mathfrak{q}$ has two images under φ^{-1} – which are near $\varphi^{-1}(a)$; consider a crosscut between these two images; the image of this crosscut under φ then defines the relevant neighborhood, which we will denote by e.g., $B(a)$. We note that i) by construction, $B(a)$ has the property that $\mathscr{J}$ enters exactly once and terminates at a, and ii) being slightly more careful if necessary to ensure the relevant crosscut is contained in $\varphi^{-1}(B_\Delta(a))$, we can also ensure that $B(a) \subset B_\Delta(a))$. Here we will consider $\eta \ll \text{dist}(a, \partial B(a))$, so that in particular, e.g., $a_n \in B(a)$.

Now let us return attention to Ω_n. We will now refurbish $\mathscr{P}$ so that it directly joins a_n to b_n and avoids all of $\partial\Omega_n$; we will call the resultant path $\mathscr{P}_r$. We claim that it is possible to draw such a $\mathscr{P}_r$ by suitably extending $\mathscr{P}$, under the above stipulations concerning $B(a)$, $B(b)$, and η. Focusing attention on $B(a)$, if this were not possible, then it must have been the case that a portion of $\mathscr{J}_n$ or a portion of $\partial\Omega_n \setminus \mathscr{J}_n$ which is approximating the other side of $\mathscr{J}$, is obstructing. This scenario implies an inner domain inside

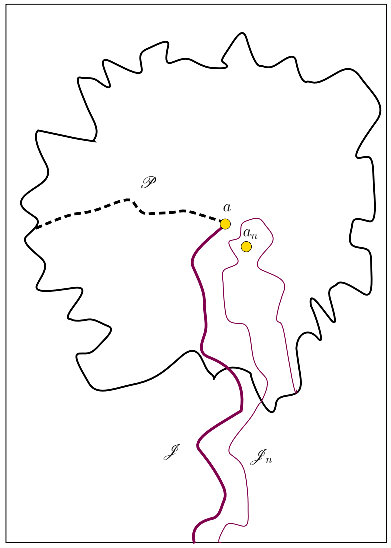

Figure 5: Failure to continue $\mathscr{P}$ to $\mathscr{P}_r$ inside $B(a)$.

$B(a)$ surrounding the tip a_n with boundary e.g., $\mathscr{J}_n$. Since $\eta \ll \text{dist}(a, \partial B(a))$, this violates sup–norm η closeness. (Here it appears that the sup–norm closeness property is crucial. For an illustration see Figure 5.)

Having achieved all this, it is again clear that the principal component of Ω_n is divided into two disjoint Jordan domains. Indeed by the fact that the approximation is well–organized, there are two circuits – both using $\mathscr{P}_r$, passing through a_n and b_n, such that one (which again is the bottom one) contains $\mathcal{C}_n$ and the other contains (the principal component of) $\Omega_n \setminus \mathcal{C}_n$, with no possibility of mixing via crisscrossing. Since $\mathscr{P}_r$ is an extension of $\mathscr{P}$, it is clear from the closing argument of 3) that w is in the bottom component and hence must be closed to $\mathcal{C}_n$ without obstruction from any portion of $\partial\Omega_n \setminus \mathcal{C}_n$.

□

5 Verification of Boundary Values for u, v, w

We are now in a position to verify boundary values for u, v, w using RSW estimates. Let us begin with a more detailed recapitulation/clarification of how we take the scaling limit of $u_\varepsilon, v_\varepsilon, w_\varepsilon$ (see [18] and [6]). Consider some exhaustion $K_n \nearrow \Omega$, with K_n compact. The RSW estimates imply equi–continuity, and hence we have $u^{(n)}_{\varepsilon_k} \to u^{(n)}$ uniformly on K_n and, at least for the models in [18] and [6],

$$F^{(n)} := u^{(n)} + e^{2\pi i/3} v^{(n)} + e^{-2\pi i/3} w^{(n)}$$

is analytic there with

$$u^{(n)} + v^{(n)} + w^{(n)} = \text{const.}$$

We may take $(\varepsilon^{(n+1)}_k) \subset (\varepsilon^{(n)}_k)$ as a subsequence which implies that

$$u^{(n+1)}_{\varepsilon_k} \to u^{(n+1)}$$

etc., so that $F^{(n+1)}$ is analytic in K_{n+1} with values agreeing with the old $F^{(n)}$ in K_n. The diagonal sequence $(u^{(n)}_{\varepsilon_n})$ converges uniformly on compact sets to some u; together with similar statements for v and w, we obtain that the limiting F is analytic on Ω. In the sequel for simplicity we will drop the (n) superscripts and just write e.g., $u_\varepsilon \to u$.

We begin with a lemma which provides us with the RSW technology which is necessary for establishing boundary values.

Lemma 5.1. *Let Ω and φ be as described. The pre–image of $\partial\Omega$ under φ is divided into a finite number of disjoint (connected) closed arcs the intersection of any adjacent pair of which is the corresponding (pre–image of the) prime end. Then for $z \in \partial\Omega \setminus \{a, b, c, \dots\}$, we identify z with a single $\varphi^{-1}(z) := \zeta$ and similarly identify its corresponding boundary component.*

(I) There exists an infinite sequence of ("square") neighborhoods (S_ℓ) centered at z such that $S_\ell \cap \Omega \neq \emptyset$ for all ℓ and $S_{\ell+1}$ is strictly contained in S_ℓ with ∂S_ℓ containing portions of the boundary component containing z and

(II) In each $S_\ell \setminus S_{\ell+1}$, there is a "yellow" circuit and/or a "blue" circuit which separates z from all other boundary components with probability that is uniformly positive as $\varepsilon \to 0$ (provided that ε is sufficiently small depending on ℓ). By separation it is meant that in the pre–image in $\mathbb{D}$, ζ, is separated from all other boundary components along any path in $\mathbb{D}$ whose image under φ tends to z.

Finally, for $z \in \{a, b, c, \dots\}$, a similar statement holds, except for the fact that here the relevant circuits separate z from all other boundary points and boundary components to which z does not belong.

Proof. Let $z \in \partial\Omega$ and let $\zeta = \varphi^{-1}(z)$ denote its corresponding pre–image. First suppose $z \notin \{a, b, c, \dots\}$ so that ζ is some finite distance from the corresponding points on $\partial\mathbb{D}$. Then consider a sufficiently small crosscut Γ of $\mathbb{D}$ surrounding ζ (a finite distance away from ζ) whose end points on $\partial\mathbb{D}$, denoted α and β, are such that α and β are in the (interior of the) boundary component of ζ. We also denote by Q the image of the interior of the region bounded by Γ and the relevant portion of $\partial\mathbb{D}$; we note that $z \in \partial Q$. Let $S_0 \subset \mathbb{C}$ be a small square centered at z whose intersection with Ω lies entirely inside Q. Then by construction, $\partial(S_0 \cap \Omega)$ can contain at most boundary pieces from the boundary component of z. Now the sequence (S_n) will be constructed similarly, with the stipulation that the linear scale of $S_{\ell+1}$ is reduced by half from that of S_ℓ.

By standard RSW estimates for the percolation problem in all of $\mathbb{C}$, there is a blue and/or yellow Harris ring inside each annulus $S_\ell \setminus S_{\ell+1}$ with probability uniformly bounded from below for ε sufficiently small. Now consider any path $\mathscr{P}$ in $\mathbb{D}$ which originates at ζ and ends outside $\varphi^{-1}(Q)$ such that the image of the path originates at z. Such a path stays in Ω and therefore must intersect the said circuit.

Identical arguments hold for $z \in \{a, b, c, \dots\}$ except for the fact that the original crosscut will now originate and end on two distinct boundary components. □

It is noted that in the presence of a circuit in $S_\ell \setminus S_{\ell+1}$, the above separation argument also applies to points in $\partial(S_m \cap Q)$ if $m \geq \ell + 1$.

Lemma 5.2 (Establishment of Boundary Values for Interior Approximations)**.** *Let Ω and φ be as described. We recall that $u_\varepsilon^B(z)$ is the probability at the ε level that there is a blue crossing from $\mathcal{A}_\varepsilon$ to $\mathcal{B}_\varepsilon$, separating z from $\mathcal{C}_\varepsilon$, and let u denote the limiting function. Then $u = 0$ on $\mathcal{C}$ in the sense that if $z_k \to z \in \mathcal{C}$ in such a way that $\varphi^{-1}(z_k) = \zeta_k \to \zeta \in \varphi^{-1}(\mathcal{C})$, then $\lim_{k\to\infty} u(z_k) = 0$. Similarly, in the vicinity of the point c, u tends to one. Analogous statements hold for v_ε^B and w_ε^B and for the yellow versions of these functions.*

Proof. Suppose a yellow Harris circuit has occurred in $S_\ell \setminus S_{\ell+1}$ and let $z_k \to z$ as described. Then, in the language of the proof of Lemma 5.1, for k sufficiently large $z_k \in Q \cap S_m$ for some $m = m(k)$ tending to ∞ as $k \to \infty$. For ε sufficiently small, it follows from (iii) in Definition 3.1 that $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ are disjoint from $(S_m \cap Q \cap \Omega_\varepsilon^\bullet)$ and since $\Omega_\varepsilon^\bullet$ is an interior approximation, the relevant portion of the circuit evidently joins with $\partial\mathcal{C}_\varepsilon$ to separate $\partial S_m \cap Q \cap \Omega_\varepsilon^\bullet$ from c, as for z as discussed near the end of the proof of Lemma 5.1. This separation would preclude the crossing event corresponding to $u_\varepsilon^B(z_k)$ since – as is clear if we look on the unit disc via the conformal map φ^{-1} – the latter necessitates (two) blue connections between the relevant portions of ∂S_m and other boundaries. Now consider k with $m(k)$ very large; then for all ε sufficiently small, the probability of at least one yellow circuit is, uniformly (in ε), close to some $p(m)$ where $p(m) \to 1$ as $m \to \infty$. It therefore follows that $u(z_k) \leq 1 - p(m(k)) \to 0$ as $z_k \to z$. Finally, boundary value of c follows the same argument: Here the blue Harris ring events accomplish the required connection between $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$. Arguments for other functions/boundaries are identical. □

Lemma 5.3 (Establishment of Boundary Values for Sup–approximations)**.** *Let Ω and φ be as described. We recall that $u_\varepsilon^B(z)$ is the probability at the ε level that there is a blue crossing from $\mathcal{A}_\varepsilon$ to $\mathcal{B}_\varepsilon$, separating* *z from $\mathcal{C}_\varepsilon$, and let u denote the limiting function. Then $u = 0$ on $\mathcal{C}$ in the sense that if $z_k \to z \in \mathcal{C}$ in such a way that $\varphi^{-1}(z_k) = \zeta_k \to \zeta \in \varphi^{-1}(\mathcal{C})$, then $\lim_{\varepsilon\to 0} u^B_\varepsilon(z_k) \to 0$. Similarly, in the vicinity of the point c, u tends to one. Analogous statements hold for v^B_ε and w^B_ε.*

Proof. We recall that we have three boundary pieces $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in counterclockwise order, where we assume without loss of generality that $\mathcal{C}$ is on the bottom. We also label the relevant marked prime ends a, b, c, in counterclockwise order, such that e.g., c is opposite to $\mathcal{C}$. Thus if we draw a path $\mathscr{P}$ between a_ε and b_ε inside Ω_ε, and $z_k \in \Omega_\varepsilon$ is inside the region formed by $\mathscr{P}$ and $\mathcal{C}_\varepsilon$ then to prevent events which contribute to u^B_ε, it is sufficient to seal z_k off from c_ε by a yellow Harris ring together with the bottom boundary $\mathcal{C}_\varepsilon$. This is precisely the setting of Lemma 4.4 (with $w = z_k$) and so we conclude that for k sufficiently large, for ε sufficiently small depending on k, in order to prevent events contributing to u^B_ε, it is indeed sufficient to seal z_k off with a yellow Harris ring.

We are now in a position to invoke Lemma 5.1. The proof follows closely as in the last part of the proof of Lemma 5.2 except for one difference: For k sufficiently large, $z_k \in S_m$ for some $m = m(k)$ which increases as k increases; however, in the case of sup–approximation, it is no longer quite so automatic that arbitrarily small Harris rings will hit the boundary $\mathcal{C}_\varepsilon$. However, given ε, we have that $\mathcal{C}_\varepsilon$ is at most a distance $\eta(\varepsilon)$ from $\mathcal{C}$, and thus, for fixed ε_0, there is some $M(\varepsilon_0)$ such that $m(k) \nearrow M(\varepsilon_0)$ as $k \to \infty$ (and $M(\varepsilon_0)) \to \infty$ as $\varepsilon_0 \to 0$). So we still have that uniformly for all $\varepsilon \leq \varepsilon_0$, $U_\varepsilon(z) \leq 1 - p(M(\varepsilon_0))$, where $p(M(\varepsilon_0))$ as before denotes the probability of at least one yellow Harris ring in the annulus $S_1 \setminus S_{M(\varepsilon_0)}$, and tends to 1 as $M(\varepsilon_0)$ tends to infinity.

□

Remark 5.4. Our arguments in fact show that the function u is continuous up to the boundary: Given any sequence $z_k \to z \in \mathcal{C}$, we have that given any $\kappa > 0$, for k sufficiently large, $|u^{(n)}_{\varepsilon_n}(z_k)| < \kappa$, uniformly in n, for n sufficiently large (or ε sufficiently small) and hence $u(z_k) < \kappa$ (c.f., the end of the proof of Theorem 5.5). We have similar statements for v and w on the corresponding boundaries.

To check that F is indeed the appropriate conformal map and thereby uniquely determine it and retrieve Cardy's Formula, we follow the arguments in [2]. We remark that while there exists certain literature on discrete complex analysis (see e.g., [10] and [7] and references therein) our situation is less straightforward since e.g., none of the functions u_N, v_N, w_N are actually discrete harmonic. Moreover, due to the fact that we are considering general domains (versus Jordan domains) and $\partial\Omega$ may not be so well–behaved, to obtain conformality requires some extra work. In any case, we will now amalgamate all ingredients to prove the following result:

Theorem 5.5. *For the models described in [6] (which includes the triangular site problem studied in [18]), consider the function $F = u + e^{2\pi i/3}v + e^{-2\pi i/3}w$, where u, v, w are the limits of $u_\varepsilon, v_\varepsilon, w_\varepsilon$. Then F is the unique conformal map between Ω and the equilateral triangle $\mathbf{T}$ with vertices at 1, $e^{2\pi i/3}$, $e^{-2\pi i/3}$.*

Proof. We claim that the following seven conditions hold:

1. F is nonconstant and analytic in Ω,
2. u, v, w (and hence F) can be continued (continuously) to $\partial\Omega$,
3. $u + v + w$ is a constant,
4. $u(c) = 1$, with similar statements for v and w at a and b,
5. $u \equiv 0$ on $\mathcal{C}$ with similar statements for v and w on $\mathcal{A}$ and $\mathcal{B}$,
6. $F \circ \varphi$ maps $\partial\mathbb{D}$ bijectively onto $\partial\mathbf{T}$,
7. $(F \circ \varphi)(\mathbb{D}) \cap (F \circ \varphi)(\partial\mathbb{D}) = \emptyset$;

from which the proposition follows immediately. Indeed, from conditions 7 and 6, $F \circ \varphi : \mathbb{D} \to \mathbf{T}$ is a conformal map (this follows directly from e.g., Theorem 4.3 in [11]). But clearly, conditions 5, 4, 3 imply that F maps Ω into $\mathbf{T}$, and further, conditions 2 and 1 imply that F maps Ω *onto* $\mathbf{T}$ (this follows from e.g., Theorem 4.1 in [11]). Altogether, conformality of F itself now follows: It is enough to show that F' never vanishes, but this follows from the fact that $0 \neq (F \circ \varphi)'(z) = F'(\varphi(z))\varphi'(z)$.

We now turn to the task of verifying conditions 1 – 7. It follows from [18], [6], and [2] that F is analytic and that $u + v + w$ is constant. On this basis, the real part of F is proportional to u plus a constant and it is seen from Lemma 5.2 (or Lemma 5.3) that u is not constant, i.e., it is close to 1 near c and close to 0 near $\mathcal{C}$. We have conditions 1 and 3. Conditions 2, 4, 5 follow from Lemma 5.2 (or Lemma 5.3) and Remark 5.4.

To demonstrate condition 7, let us write $\mathrm{Re}(F) = (3/2)u - 1/2$. Then if we show that $u \neq 0$ in Ω, then we have demonstrated that $F(\Omega)$ does not intersect $F(\mathcal{C})$. The latter follows since once $z \in \Omega$, we can construct a tube of bounded conformal modulus connecting $\mathcal{A}$ to $\mathcal{B}$ going underneath z, and within this tube, by standard percolation arguments which go back to [1], we can construct a monochrome path separating z from $\mathcal{C}$. Condition 6 follows in a similar spirit: E.g., on the $\mathcal{A}$ boundary, if $z \neq q$, but $|z - q| \ll 1$, then by the argument of Lemma 5.1, $u(z)$ is close to $u(q)$ (since both can be surrounded by many annuli in which e.g., a blue circuit occurs). Similar arguments for v and w and other boundaries directly imply continuity of all functions on all boundaries of Ω. Moreover, this implies, e.g., $u \circ \varphi^{-1}(\mathcal{A})$ is continuous on the relevant portion of the circle starting (at $\varphi^{-1}(c)$) with the value 1 and ending (at $\varphi^{-1}(b)$) with the value 0 and thus achieving all values in $[0, 1]$. Similarly statements hold for the other functions on the other boundaries. Condition 6 now follows directly.

□

Remark 5.6. It is worth noting that while using only arguments involving RSW bounds, we have determined that 1) the u, v, w's can be continued to the boundary and 2) partial boundary values, e.g., $u \equiv 0$ on $\mathcal{C}$, sufficient determination of boundary values requires additional ingredients. In particular, we also needed that e.g., $v + w \equiv 1$ on $\mathcal{C}$; this would follow from $u + v + w \equiv 1$ which at present seems only to be derivable from analyticity considerations. Duality implies e.g., $v_\varepsilon^B + w_\varepsilon^Y \equiv 1$ on $\mathcal{C}$, but we cannot go any further without color symmetry as in the site percolation on the triangular lattice case ([18]) or some (asymptotic) color symmetry restoration as was established for the models in [6].

Definition 5.7. Let us define, as in [3], $C_\varepsilon(\Omega_\varepsilon, a_\varepsilon, b_\varepsilon, c_\varepsilon, d_\varepsilon)$ to be the crossing probability of the rectangle $(\Omega_\varepsilon, a_\varepsilon, b_\varepsilon, c_\varepsilon, d_\varepsilon)$ with percolation also taking place at the ε–scale.

Next consider the (unique) conformal map which takes (Ω, a, b, c, d) to $(\mathbb{H}, 1 - x, 1, \infty, 0)$, where, clearly, $0 < x < 1$ and $x = x(\Omega, a, b, c, d)$. Then we denote by $C_0(\Omega, a, b, c, d)$ the function

$$\frac{\int_0^x (s(1-s))^{-2/3}\, ds}{\int_0^1 (s(1-s))^{-2/3}\, ds}, \tag{1}$$

i.e., Cardy's Formula.

Let us recall/observe that $C_0(\Omega, a, b, c, d)$ is equal to e.g., $u(d)$ with $d \in \mathcal{A}$ (see Section 1). We now have

Theorem 5.8. *For the models described in [6] with the assumption $M(\partial\Omega) < 2$ (which includes the triangular site problem studied in [18], where the assumption on $\partial\Omega$ is unnecessary) Cardy's Formula can be established via an interior or sup–approximation, i.e.,*

$$C_\varepsilon(\Omega_\varepsilon, a_\varepsilon, b_\varepsilon, c_\varepsilon, d_\varepsilon) \to C_0(\Omega, a, b, c, d)$$

if (Ω_ε) is an interior or sup–approximation to Ω.

Proof. For the site percolation model, this follows from [18], [6], [2], and Theorem 5.5. For the model described in [6], the interior analyticity statement in sufficient generality is verified in [3], §4.4. □

Finally, let us single out the cases that will be used in the proof of the Main Theorem in [3].

Corollary 5.9. *Consider the models described in [6] (which includes the triangular site problem studied in [18]) on a bounded domain* Ω *with boundary Minkowski dimension less than two (if necessary) and two marked boundary points* a *and* c. *Suppose we have* $\mathbb{X}^{\varepsilon}_{[0,t]} \to \mathbb{X}_{[0,t]}$ *in the* **Dist** *norm where* $\mathbb{X}^{\varepsilon}_{[0,t]}$ *is the trace of a discrete Exploration Process starting at* a *and aiming towards* c, *stopped at some time* t, *then*

$$C_{\varepsilon}(\Omega_{\varepsilon} \setminus \mathbb{X}^{\varepsilon}_{[0,t]}, \mathbb{X}^{\varepsilon}_{t}, b_{\varepsilon}, c_{\varepsilon}, d_{\varepsilon}) \to C_0(\Omega \setminus \mathbb{X}_{[0,t]}, \mathbb{X}_t, b, c, d).$$

Further, it is possible to extract a slightly stronger statement which will be used in the proof of the Main Theorem in [3]. For the sake of [3] we will state these results in the **Dist** norm (c.f., Remark 4.1). For purposes of clarity, we first state a lemma:

Proposition 5.10. *Let us denote the type of (slit) domain under consideration by* Ω^{γ} *and abbreviate, by abuse of notation, e.g.,* $C_{\varepsilon}((\Omega^{\gamma})_{\varepsilon}) := C_{\varepsilon}(\Omega_{\varepsilon} \setminus \gamma_{\varepsilon}([0,t]), \gamma_{\varepsilon}(t), b_{\varepsilon}, c_{\varepsilon}, d_{\varepsilon})$ *(but here,* γ *could stand for other boundary pieces as detailed in Definition 4.2). Then for any sequence* $\gamma_n \to \gamma$ *in the* **Dist** *norm and any sequence* (ε_m) *converging to zero,*

$$\lim_{n,m\to\infty} C_{\varepsilon_m}\left[(\Omega^{\gamma_n})_{\varepsilon_m}\right] = C_0(\Omega^{\gamma}),$$

regardless of how n *and* m *tend to infinity. Here all approximations are sup–approximations.*

Proof. From Lemma 5.3 we have that e.g., if $\gamma^{(n)}_{\varepsilon_m} \to \gamma_n$ is any sup–approximation, then $C_{\varepsilon_m}[(\Omega^{\gamma_n})_{\varepsilon_m}] \to C_0(\Omega^{\gamma_n})$. The result follows by noting that $\gamma^{(n)}_{\varepsilon_m}$ is also a sup–approximation to γ as *both* $m, n \to \infty$. We emphasize that the reason for such robustness of Lemma 5.3 is because the proof is completely insensitive to how γ_{ε} converges to γ as $\varepsilon \to 0$. All that is needed is that γ_{ε} is sufficiently close to γ and ε is sufficiently small, which is inevitable if ε is tending to zero and γ_{ε} is tending to γ. □

Corollary 5.11. *Considered the models described in [6] (which includes the triangular site problem studied in [18]) on a bounded domain* Ω *with boundary Minkowski dimension less than two (if necessary) and two marked boundary points* a *and* c. *Consider* $\mathscr{C}_{a,c,\Delta}$, *the set of Löwner curves which begin at* a, *are aiming towards* c *but have not yet entered the* Δ *neighborhood of* c *for some* $\Delta > 0$. *Suppose we have* $\gamma_{\varepsilon} \to \gamma$ *e.g., in the* **Dist** *norm, then*

$$C_{\varepsilon}(\Omega_{\varepsilon} \setminus \gamma_{\varepsilon}([0,t]), \gamma_{\varepsilon}(t), b_{\varepsilon}, c_{\varepsilon}, d_{\varepsilon}) \to C_0(\Omega \setminus \gamma([0,t]), \gamma(t), b, c, d)$$

pointwise equicontinuously in the sense that

$$\begin{gathered} \forall \kappa > 0, \quad \forall \gamma \in \mathscr{C}_{a,c,\Omega}, \quad \exists \delta(\gamma) > 0, \quad \exists \varepsilon_{\gamma}, \\ \text{such that} \\ \forall \gamma' \in \mathcal{B}_{\delta(\gamma)}(\gamma), \quad \forall \varepsilon \le \varepsilon_{\gamma}, \\ |C_{\varepsilon}((\Omega \setminus \gamma)_{\varepsilon}([0,t])), (\gamma(t))_{\varepsilon}, b_{\varepsilon}, c_{\varepsilon}, d_{\varepsilon}) - C_{\varepsilon}((\Omega \setminus \gamma')_{\varepsilon}([0,t])), (\gamma'(t))_{\varepsilon}, b_{\varepsilon}, c_{\varepsilon}, d_{\varepsilon})| < \kappa. \end{gathered} \tag{2}$$

Here $B_{\delta}(\gamma)$ *denotes the* **Dist** *neighborhood of* γ.

Proof. This is immediate from Proposition 5.10. Negation of the conclusion in the statement means that there exists a sequence $\gamma_n \to \gamma$ and $\varepsilon_n \to 0$ such that $|C_{\varepsilon_n}((\Omega^{\gamma_n})_{\varepsilon_n}) - C_{\varepsilon}((\Omega^{\gamma})_{\varepsilon_n})| > \kappa > 0$ for all ε_n, which clearly contradicts the fact that both of these objects converge to the limit $C_0(\Omega^{\gamma})$.

□

Remark 5.12. We remark that (2) holds even if "$\varepsilon = 0$" and thus implies continuity of Cardy's Formula in the "**Dist** norm". However, we note that Lemma 5.3, being merely a limiting statement, would be highly inadequate if one had in mind some uniformity of the convergence or uniformity of the continuity.

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